INDEX TO VOLUME 61

AUTHORS

TITLES

PROBLEMS

P, S, Q, *and* C *refer to Proposals, Solutions, Quickies, and Comments, respectively. Page numbers are given in parentheses. Thus,* P1287(46) *refers to proposal number* 1287 *which appears on page* 46.

NEWS & LETTERS

Vol. 61 No. 5 December 1988

MATHEMATICS MAGAZINE

- Coding Theory and a Problem of Ulam
- Palindromes, Density and Divisibility
- When Does the Symmetry Property Hold?

An Official Publication of The MATHEMATICAL ASSOCIATION OF AMERICA

EDITORIAL POLICY

The aim of *Mathematics Magazine* is to provide lively and appealing mathematical exposition. This is not a research journal and, in general, the terse style appropriate for such a journal (lemma-theorem-proof-corollary) is not appropriate for an article for the *Magazine*. Articles should include examples, applications, historical background, and illustrations, where appropriate. They should be attractive and accessible to undergraduates and would, ideally, be helpful in supplementing undergraduate courses or in stimulating student investigations. Articles on pedagogy alone, unaccompanied by interesting mathematics, are not suitable. Neither are articles consisting mainly of computer programs unless these are essential to the presentation of some good mathematics. Manuscripts on history are especially welcome, as are those showing relationships between various branches of mathematics and between mathematics and other disciplines.

The full statement of editorial policy appears in this *Magazine*, Vol. 54, pp. 44–45, and is available from the Editor. Manuscripts to be submitted should not be concurrently submitted to, accepted for publication by, nor published by another journal or publisher.

Send new manuscripts to: G. L. Alexanderson, Editor, Mathematics Magazine, Santa Clara University, Santa Clara, CA 95053. Manuscripts should be typewritten and double spaced and prepared in a style consistent with the format of *Mathematics Magazine*. Authors should submit the original and one copy and keep one copy. Illustrations should be carefully prepared on separate sheets in black ink, the original without lettering and two copies with lettering added.

AUTHORS

Ivan Niven received his Ph.D. from the University of Chicago, working with L. E. Dickson on aspects of the Waring problem in Diophantine equations. His early schooling was in Canada, with undergraduate work at the University of British Columbia. Following a postdoctoral fellowship year with Hans Rademacher, and service on the faculties of the University of Illinois and Purdue University, he went to the University of Oregon, where he is now professor emeritus. His short proof of the irrationality of π, based on calculus, in the 1947 Bulletin of the AMS, received much attention. He feels that his best mathematical work was this: Herman Weyl had established that if $\beta, 2\beta, 3\beta, \ldots$ are the multiples of any irrational number β, then the fractional parts are uniformly distributed in the unit interval. Niven proved that the integral parts, $[\beta], [2\beta], [3\beta], \ldots$ are also uniformly distributed, in a different sense, of course. This 1961 paper in the *Transactions of the AMS* launched a new area of study in number theory: uniformly distributed sequences of integers. He was the Earl Raymond Hedrick lecturer of the MAA in 1960, speaking on topics in Diophantine approximation theory. His paper, "Formal Power Series," won the Lester R. Ford Award in 1970. Although his interest was primarily in research in the first twenty-five years or so following his Ph.D. in 1938, he has turned more recently to expository writing. Five of his seven books are still in print; the most recent is *Maxima and Minima Without Calculus*, published in the Dolciani series of the MAA in 1981. The book *Introduction to the Theory of Numbers*, written jointly with H. S. Zuckerman, is now being revised for a fifth edition, with Hugh L. Montogomery of the University of Michigan as a new co-author. He was the President of the MAA in 1983–84.

Peter Hilton spent his war service during WWII deciphering high-grade German signals at Bletchley Park. Among the mathematicians he worked with was the great British topologist J.H. C. (Henry) Whitehead. After the war, Whitehead invited Peter to become his research student at Oxford University, an invitation Peter accepted entirely on the strength of his friendship with Whitehead and respect for his obviously remarkable talent, since Peter then had very little idea what topology was. However, this fortunate association enabled Peter to meet, while still a young student, many of the great topologists of the immediate postwar period—Hopf, Eckmann, Eilenberg, Mac Lane, Steenrod, Alexandroff, among others—and learn much of the pioneering days of algebraic topology. Unfortunately, Henry Whitehead's wonderfully creative life came to an untimely end in 1960, when he was only 55 years old.

Acknowledgements. The quotations on p. 313 were provided by Harold P. Boas and Ralph P. Boas, respectively.

Cover. Heinz Hopf (See Hilton article on p. 282.)

Vol. 61 No. 5 December 1988

MATHEMATICS MAGAZINE

The *MATHEMATICS MAGAZINE* (ISSN 0025-570X) is published by the Mathematical Association of America at 1529 Eighteenth Street, N.W., Washington, D.C. 20036 and Montpelier, VT, bimonthly except July/August.
The annual subscription price for the MATHEMATICS MAGAZINE to an individual member of the Association is $11 included as part of the annual dues. (Annual dues for regular members, exclusive of annual subscription prices for MAA journals, are $22. Student, unemployed and emeritus members receive a 50% discount; new members receive a 30% dues discount for the first two years of membership.) The nonmember/library subscription price is $28 per year. Bulk subscriptions (5 or more copies) are available to colleges and universities for classroom distribution to undergraduate students at a 41% discount ($6.50 per copy—minimum order $32.50).
Subscription correspondence and notice of change of address should be sent to the Membership/Subscriptions Department, Mathematical Association of America, 1529 Eighteenth Street, N.W., Washington, D.C. 20036. Microfilmed issues may be obtained from University Microfilms International, Serials Bid Coordinator, 300 North Zeeb Road, Ann Arbor, MI 48106.
Advertising correspondence should be addressed to Ms. Elaine Pedreira, Advertising Manager, The Mathematical Association of America, 1529 Eighteenth Street, N.W., Washington, D.C. 20036.

Second class postage paid at Washington, D.C. and additional mailing offices.
Postmaster: Send address changes to Mathematics Magazine Membership/Subscriptions Department, Mathematical Association of America, 1529 Eighteenth Street, N.W., Washington, D.C. 20077-9564.
PRINTED IN THE UNITED STATES OF AMERICA

ARTICLES

Coding Theory Applied to a Problem of Ulam

IVAN NIVEN
University of Oregon
Eugene, OR 97403

In this paper the theory of error-correcting codes is applied to a problem in mathematics. We add at once that no background is needed in computer science to understand the argument. The problem under consideration was stated by S. M. Ulam [1] in this way:

> Someone thinks of a number between one and one million (which is just less than 2^{20}). Another person is allowed to ask up to twenty questions, to which the first person is supposed to answer only yes or no. Obviously the number can be guessed by asking first: Is the number in the first half-million? and then again reduce the reservoir of numbers in the next question by one-half, and so on. Finally the number is obtained in less than $\log_2(1000000)$. Now suppose one were allowed to lie once or twice, then how many questions would one need to get the right answer?

In an article in this journal [2], Joel Spencer discussed the Ulam problem in the case where the responder to the questions is allowed, but not required, to give one false answer. (This basic assumption applies throughout the present paper.) Spencer narrowed the outcome down to two possibilities, and remarked in conclusion, "But it seems very difficult to determine whether the answer to Ulam's original problem is twenty-five or twenty-six." We show here that twenty-five questions will suffice, using quite a different method.

Consider now the generalization of the Ulam problem from 10^6 to any positive integer n: the responder thinks of an integer in the set $\{1, 2, \ldots, n\}$ and the questioner wants to determine this special integer. If the responder is allowed, but not required, to give just one false answer, what is the minimum number of questions needed? Andrzej Pelc [9] has solved this problem; denoting the minimum number by $g(n)$ we give Pelc's value later in this paper.

Following error-correcting code theory, we have a different method that gets the answer with $f(n)$ questions, where $f(n) = g(n)$ for some values of n, but $f(n) = g(n) + 1$ for others. At the cost, then, of perhaps one additional question we get a very simple algorithm that points directly at the special integer. Furthermore, the $f(n)$ questions can be formulated once and for all with no feedback needed, so that there are no instructions of the sort, "If the answer is yes, proceed this way; if no, proceed that way." In passing we note that $f(10^6) = g(10^6) = 25$. Also $f(n) = g(n)$ in 150 of the cases from $n = 2$ to $n = 250$. If n is a power of 2, $f(n) = g(n)$.

The plan of the paper is to start with an easy example to illustrate the method. Then the Ulam problem is addressed, with the algorithm stated after the contrived notation is set forth. After an example is given to illustrate the procedure, the method is generalized from 10^6 to the first n natural numbers in Section 7. In Section 8 it is proved that $f(n) = g(n)$ or $f(n) = g(n) + 1$.

An Easy Example

In item #34 in a book of problems by Donald J. Newman [8], we are asked to determine an integer x in the set $\{0, 1, 2, \ldots, 15\}$ by asking seven questions, where the responder may give one false answer. The following solution differs from that of Newman in that seven questions are stated once and for all, with no feedback needed. Let the 16 integers be written in binary form, each with four digits, thus $0000, 0001, \ldots, 1111$. Denote the digits of the unknown integer x by b_1, b_2, b_3, b_4, and define b_5, b_6 and b_7 by

$$b_5 = b_1 + b_2 + b_3, \qquad b_6 = b_1 + b_2 + b_4, \qquad b_7 = b_1 + b_3 + b_4$$

using binary addition. The seven questions are: "Is $b_j = 0$?" for $j = 1, 2, \ldots, 7$. If the answer to the jth question is yes, define $c_j = 0$; otherwise define $c_j = 1$. The seven digits $c_1, c_2, \ldots, c_7$ are, apart from one possible exception, the same as $b_1, b_2, \ldots, b_7$. We attach the label C for consistent to c_5 if $c_5 = c_1 + c_2 + c_3$ holds; otherwise attach the label I for inconsistent. Similarly, attach the label C or I to each of c_6 and c_7 according as $c_6 = c_1 + c_2 + c_4$ holds modulo 2 in the first case and $c_7 = c_1 + c_3 + c_4$ in the second. For example, if $c_1, c_2, \ldots, c_7$ are $1, 0, 1, 1, 0, 1, 0$ then c_5, c_6, c_7 are labelled C, I, I. Simple logic reveals that with this pattern $c_4 \neq b_4$. In the example we see that the binary digits of x are 1010, and hence $x = 10$ in decimal notation.

There are 8 possible patterns for c_5, c_6, c_7, namely,

CCC, CCI, CIC, ICC, CII, ICI, IIC, III.

In the first case here all questions have been answered correctly. In the other seven cases, a false answer has been given to question #7, #6, #5, #4, #3, #2, #1 respectively. Thus the binary digits of x are easily determined in all cases.

The Ulam Problem

As might be expected, the notation is slightly more complicated for larger values of n as contrasted with the case $n = 16$ above. We note that the integer 10^6 has twenty digits when written to base two, because $2^{19} < 10^6 < 2^{20}$. Hence any integer from 1 to 10^6 may be written in binary notation with twenty digits, each zero or one. The responder to our questions is thinking of an integer x between 1 and 10^6, say with binary digits

$$a_1, a_2, a_3, \ldots, a_{20}. \tag{1}$$

For example, if $x = 37$ then $a_{15} = a_{18} = a_{20} = 1$, and all the other a's are zero. We shall determine these twenty digits by asking 25 questions of the yes or no type, allowing for the possibility that one of these questions is answered incorrectly. It is preferable to write the twenty digits (1) with a different notation, namely,

$$b_3, b_5, b_6, b_7, b_9, b_{10}, b_{11}, b_{12}, b_{13}, b_{14}, b_{15}, b_{17},$$
$$b_{18}, b_{19}, b_{20}, b_{21}, b_{22}, b_{23}, b_{24}, b_{25}. \tag{2}$$

The subscripts 1, 2, 4, 8, and 16 are not present here. For them we define the error-correcting digits $b_1, b_2, b_4, b_8, b_{16}$, each 0 or 1, by the binary equations

$$b_1 = b_3 + b_5 + b_7 + b_9 + b_{11} + b_{13} + b_{15} + b_{17} + b_{19} + b_{21} + b_{23} + b_{25}, \quad (3)$$

$$b_2 = b_3 + b_6 + b_7 + b_{10} + b_{11} + b_{14} + b_{15} + b_{18} + b_{19} + b_{22} + b_{23}, \quad (4)$$

$$b_4 = b_5 + b_6 + b_7 + b_{12} + b_{13} + b_{14} + b_{15} + b_{20} + b_{21} + b_{22} + b_{23}, \quad (5)$$

$$b_8 = b_9 + b_{10} + b_{11} + b_{12} + b_{13} + b_{14} + b_{15} + b_{24} + b_{25}, \quad (6)$$

$$b_{16} = b_{17} + b_{18} + b_{19} + b_{20} + b_{21} + b_{22} + b_{23} + b_{24} + b_{25}. \quad (7)$$

There is nothing random about these equations. The subscripts in (3) comprise all the odd integers from 1 to 25 having "1" as the far right digit when written in binary form. The subscripts in (4) are all the integers from 1 to 25 having "1" in the second position from the right when written in binary form. Similarly, the subscripts in equations (5), (6) and (7) are all the integers from 1 to 25 having "1" in the third, fourth, and fifth positions from the right, respectively.

The Questions

Having informed the responder of the definitions above, we ask 25 questions, where the jth question asks "Is $b_j = 1$?", for $j = 1, 2, 3, \ldots, 25$. If the answer is yes, we write $c_j = 1$, whereas we define $c_j = 0$ if the answer is no. Note that if no question is answered falsely, then $b_j = c_j$ for all $j = 1, 2, \ldots, 25$. But if the responder answers one of the questions falsely, say the kth question, then $b_j = c_j$ in all cases except one, where $b_k = c_k + 1$ in binary notation. Note also that whereas the responder knows the values of the a's, b's and c's, we know only the values of the c's. From this information we can deduce the values of the b's and a's, and hence the value of x, the unknown integer.

The Algorithm

We state the algorithm, followed by a detailed example, and then we justify the algorithm. For convenience let S_1, S_2, S_4, S_8, and S_{16} denote respectively the sets of coefficients in equations (3), (4), (5), (6), and (7): for example, $S_2 = \{2, 3, 6, 7, 10, 11, 14, 15, 18, 19, 22, 23\}$. We now define the key binary digits, p_{16}, p_8, p_4, p_2, and p_1. First define

$$p_{16} = \sum c_j, \text{ summed over all } j \text{ in } S_{16}. \quad (8)$$

Similarly, define p_8, p_4, p_2, and p_1 to be $\sum c_j$ where the sums are to be taken over all j in S_8, S_4, S_2 and S_1, respectively.

Consider the five digit integer in the binary system, $p_{16}p_8p_4p_2p_1$. This is not an algebraic product, but a five digit integer to base two. Let k be the decimal representation of this integer, so that

$$k = 16p_{16} + 8p_8 + 4p_4 + 2p_2 + p_1. \quad (9)$$

This equation is not binary; k may be larger than 1.

If $k = 0$, that is, if $p_{16} = p_8 = p_4 = p_2 = p_1 = 0$, then no question has been answered falsely, so that $c_j = b_j$ for $j = 1, 2, \ldots, 25$. In this case the integer x in binary form is

$$x = c_3c_5c_6c_7c_9c_{10}c_{11}c_{12}c_{13}c_{14}c_{15}c_{17}c_{18}c_{19}c_{20}c_{21}c_{22}c_{23}c_{24}c_{25}. \quad (10)$$

This is not an algebraic product, but an integer to base two with twenty digits.

If $k>0$ then c_k is the troublemaker in the sense that $c_k \neq b_k$. In this case the integer x in binary form is (10) with perhaps one change: replace c_k by $1+c_k$, modulo two. We say "perhaps" one change, because if k is any of 1, 2, 4, 8, or 16, then no change is needed in (10).

An Example

Consider the case where the digits $c_1, c_2, \dots, c_{25}$ are

$$0,0,1,0,1, \quad 0,0,0,0,1, \quad 0,0,1,1,1, \quad 1,0,1,1,0,1,1,0,1,1,$$

in that order. From equation (8) and the analogs for p_8 etc. we get $p_{16}=1$, $p_8=0$, $p_4=0$, $p_2=1$, and $p_1=1$. From (9) we have $k=16+2+1=19$. Thus c_{19} is the troublemaker; since $c_{19}=1$ we know that $b_{19}=0$. By (10) we note that x in binary form is $11,000,100,111,010,011,011$. Thus in decimal form the integer we seek is 806,555.

Why Does it Work?

First, if $b_j=c_j$ for every j in S_{16}, then $p_{16}=0$ because of equations (7) and (8). But if $b_j \neq c_j$ for some j in S_{16}, then $p_{16}=1$. Thus $p_{16}=0$ if and only if $b_j=c_j$ for every j in S_{16}. Analogous results hold for p_8, p_4, p_2, and p_1. It follows that $p_{16}=p_8=p_4=p_2=p_1=0$ if and only if $b_j=c_j$ for every $j=1,2,\dots,25$. This explains the first part of the algorithm, because if all questions are answered correctly the digits in (10) are the same as those in (2).

On the other hand, if one or more of p_{16}, p_8, p_4, p_2, and p_1 has value 1, then for some h we know that $c_h \neq b_h$. Now h is one of the numbers $1,2,3,\dots,25$, and so can be written with five digits in binary form. These five digits are identified one by one by p_{16}, p_8, p_4, p_2, and p_1 because equation (8) contains c_h if and only if h belongs to S_{16}, and similarly for the analogs to equation (8) for p_8, p_4, p_2 and p_1. Thus h is tracked down, digit by digit, in the calculations of the p_i.

The General Case

In generalizing the problem from the Ulam special case with 10^6 to any natural number n, we start by defining some basic sets of positive integers. Denote by N the sequence $3,5,6,7,9,10,\dots$ of natural numbers with the powers of 2 omitted. We will need some notation for the mth term in this sequence: it is $m+r$, where r is the least positive integer satisfying

$$m+r<2^r. \tag{11}$$

To see this we note that $m+(r-1) \geq 2^{r-1}$ or $m+r>2^{r-1}$. Thus r could have been defined as the unique positive integer satisfying $2^{r-1}<m+r<2^r$. This shows that there are exactly r powers of 2 less than $m+r$, including 2^0.

Next we define the (overlapping) sets T_{2^j} for $j=0,1,2,\dots$. Every natural number has a unique representation as a sum of distinct terms from the set $\{1,2,4,8,16,\dots\}$. If 2^j is included in that representation of a natural number t, then t is included in T_{2^j}; otherwise t is excluded. For example, T_1 is just the set of positive odd integers. In fact

T_1, T_2, T_4, T_8 and T_{16} are just S_1, S_2, S_4, S_8 and S_{16} from the Ulam problem extended to be infinite sets.

Now consider the problem of finding an unknown x in the first n natural numbers under the same conditions as before, that one false answer may be given by the responder. We switch from $\{1,2,3,\dots,n\}$ to $\{0,1,2,\dots,n-1\}$ to minimize the number of binary digits; this matters only if n is a power of 2. Define m as the number of digits in $n-1$ in binary form, so that

$$m = 1 + [\log_2(n-1)]. \tag{12}$$

where the square brackets denote the greatest integer function. We now prove that we can determine x with $m+r$ questions, with r defined as in (11).

Write every integer from 0 to $n-1$ in binary form using m digits. Suppose the binary digits of the unknown x are

$$b_3, b_5, b_6, b_7, b_9, \dots, b_{m+r}, \tag{13}$$

where the subscripts are the first m integers in the set N, the natural numbers with the powers of 2 omitted. This generalizes the list (2) of 20 digits in the Ulam problem above, and the rest of that discussion can now be extended quite easily. For $j = 0, 1, 2, \dots, r-1$ we use binary addition to define

$$b_{2^j} = \sum b_i, \qquad 2^j < i \leqq m+r, \qquad i \in T_{2^j}, \tag{14}$$

where the sum is subject to the given conditions on i.

The $m+r$ questions are "What is b_i?", with $i = 1, 2, \dots, m+r$. As in the Ulam problem, we define $c_i = 1$ if the answer to the ith question is YES, and $c_i = 0$ if the answer is NO. For $j = 0, 1, 2, \dots, r-1$, use binary addition to define

$$p_{2^j} = \sum c_i, \qquad 1 \leqq i \leqq m+r, \qquad i \in T_{2^j}. \tag{15}$$

Then we set

$$k = p_1 + 2p_2 + 4p_4 + 8p_8 + \cdots + wp_w, \text{ with } w = 2^{r-1},$$

where the addition is not binary, but in the ordinary ring of integers. If $k = 0$ then $b_i = c_i$ for all i. If $k \neq 0$ then $b_i = c_i$ in all cases except when $i = k$. Thus x is readily determined. The explanation why the scheme works has been given in detail in the Ulam problem. The idea is that the digits $p_1, p_2, p_4, \dots$ have been defined so as to be the binary digits (from right to left) of the incorrectly answered question.

The algorithm above uses say $f(n)$ questions, where $f(n) = m + r$, with m defined in (12) and r in (11). It is not difficult to prove that $f(n)$ can be expressed in terms of m alone,

$$f(n) = m + r = m + 1 + [\log_2(m + 1 + \log_2 m)]. \tag{16}$$

One way to establish this is to verify it for m in the set

$$\{2^t - t, 2^t - t + 1, 2^t - t + 2, \dots, 2^{t+1} - t - 2\}. \tag{17}$$

For each of these values of m, the value of r from (11) is readily calculated to be $t+1$, and the final expression in (16) is in harmony with this. Moreover, every positive integer turns up in the set (17) if we take $t = 1, 2, 3, \dots$ successively.

However, although the algorithm is easily understood and applied, it does not give the minimum number of questions for all values of n. The minimum number of questions turns out to be $f(n)$ or $f(n)-1$.

The Minimum Number of Questions

Let $g(n)$ denote the minimum number of questions needed to find an unknown x in $\{1,2,\ldots,n\}$, where the responder is allowed but not required to give one false answer. The minimum number $g(n)$ has been determined by Andrzej Pelc in a recent paper [9]. It is the least positive integer h satisfying

$$\begin{cases} n(h+1) \leqq 2^h \text{ if } n \text{ is even, and} \\ \\ n(h+1)+(h-1) \leqq 2^h \text{ if } n \text{ is odd.} \end{cases} \tag{18}$$

It is easy to calculate $g(n)$ for any special n from these formulas. For example, $g(16)=f(16)=7$ (Newman problem), $g(10^6)=f(10^6)=25$ (Ulam problem), $g(10^9)=f(10^9)=36$, $g(50)=f(50)-1=9$.

It is readily proved that $f(n)$ is the minimum number of questions needed when n is a power of 2, say $n=2^m$. One way is to use a standard argument from the theory of single-error-correcting codes; see Thompson [3; pp. 20, 21]. But it is more convenient to use the Pelc formula (18), since it is at hand. Thus $g(2^m)$ is the least positive integer h such that

$$2^m(h+1) \leqq 2^h \quad \text{or} \quad h+1 \leqq 2^{h-m} \quad \text{or} \quad h < 2^{h-m}.$$

This is the same as the least $m+r$ satisfying $m+r<2^r$, as in (11).

Finally, we establish

$$f(n)=g(n) \text{ or } f(n)=g(n)+1 \tag{19}$$

for all values of n. We know that $f(2^m)=g(2^m)$ and from (11) we can calculate $f(2^{m+1})-f(2^m)=1$ or 2. In the first of these cases, for any n strictly between 2^m and 2^{m+1} we see that either $g(n)=f(2^m)$ or $g(n)=f(2^{m+1})=1+f(2^m)$. Also $f(n)=f(2^{m+1})$, and (19) follows.

The second case, $f(2^{m+1})-f(2^m)=2$, occurs if and only if $m+r+1=2^r$, where r is the least integer satisfying $m+r<2^r$, as in (11). For any n strictly between 2^m and 2^{m+1} we have $f(n)=f(2^{m+1})=f(2^m)+2=2^r+1$, and Pelc's formulas (18) give $g(2^m+1)=2^r$ unless $r=2$. Hence (19) follows because if $r=2$ we have $m=1$, and in this case $f(3)=g(3)=5$.

Further Comments

In his approach to the problem, Spencer [2] imposes a very strong limitation on the questions allowed, namely that "all questions must be of the form: Is $x<a$?" For example, such a simple question as "Is x even?" is ruled out. This goes beyond Ulam's requirement which we have followed, as also does Pelc, that the questions must be answerable by a simple yes or no. Thus the problem addressed by Spencer may not be equivalent to the Ulam problem.

It is a simple observation that the binary search procedure (in the no-false-answer situation) can be formulated without feedback. If the responder tranlates the number

x between 1 and 10^6 into binary notation with twenty digits, we can determine these digits one at a time by asking "Is the jth digit zero?" for $j = 1, 2, \ldots, 20$. These questions can be asked even with x in its decimal system representation; the jth question becomes "Is x in the set S_j?" using appropriately constructed sets of integers $S_1, S_2, \ldots, S_{20}$.

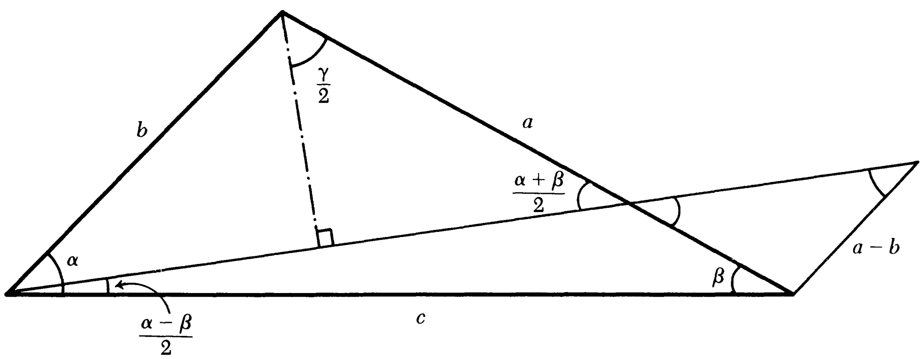
The method used in this paper has been adapted from the theory of error-correcting codes, originated by R. W. Hamming and others. In that theory an error is not deliberately introduced as in the Ulam problem, but may creep into the data in other ways, and the computer can be programmed to detect the error and correct it. Very accessible introductions to this theory have been given by Thompson [3] and Pless [4]. For a fuller statement, see MacWilliams and Sloane [5].

REFERENCES

1. S. M. Ulam, *Adventures of a Mathematician*, Scribner, New York, 1976, p. 281.
2. Joel Spencer, Guess a number—with lying, this MAGAZINE 57 (1984), 105–108.
3. Thomas M. Thompson, *From Error-Correcting Codes through Sphere Packings to Simple Groups*, Carus Monograph #21, Mathematical Association of America, Washington, D.C., 1983, especially pp. 19–26.
4. Vera Pless, *Introduction to the Theory of Error-Correcting Codes*, John Wiley, New York, 1982.
5. F. J. MacWilliams and N. J. A. Sloane, *The Theory of Error-Correcting Codes*, North-Holland Pub. Co., 1977.
6. D. J. Kleitman, A. R. Meyer, R. L. Rivest, J. Spencer, K. Winklmann, Coping with errors in binary search procedures, *J. Comput. System Sci.* 20 (1980), 396–404.
7. T. A. Brylawski, *The Mathematics of Watergate (An Analysis of a Two-Party Game)*, Univ. of N. Carolina, 1978.
8. Donald J. Newman, *A Problem Seminar*, Springer-Verlag, New York, 1980.
9. Andrzej Pelc, Solution of Ulam's problem on searching with a lie, *J. of Combin. Theory* Ser. A, 44 (1987) 129–42.

Proof Without Words: Mollweide's Equation

$$(a-b)\cos\frac{\gamma}{2} = c\sin\left(\frac{\alpha}{2}-\frac{\beta}{2}\right)$$



—H. ARTHUR DEKLEINE
California Polytechnic State University
San Luis Obispo, CA 93407

x between 1 and 10^6 into binary notation with twenty digits, we can determine these digits one at a time by asking "Is the jth digit zero?" for $j = 1, 2, \dots, 20$. These questions can be asked even with x in its decimal system representation; the jth question becomes "Is x in the set S_j?" using appropriately constructed sets of integers $S_1, S_2, \dots, S_{20}$.

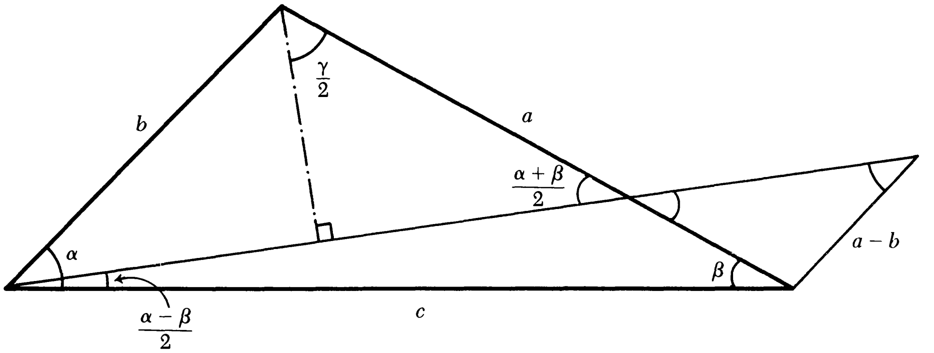
The method used in this paper has been adapted from the theory of error-correcting codes, originated by R. W. Hamming and others. In that theory an error is not deliberately introduced as in the Ulam problem, but may creep into the data in other ways, and the computer can be programmed to detect the error and correct it. Very accessible introductions to this theory have been given by Thompson [3] and Pless [4]. For a fuller statement, see MacWilliams and Sloane [5].

REFERENCES

1. S. M. Ulam, *Adventures of a Mathematician*, Scribner, New York, 1976, p. 281.
2. Joel Spencer, Guess a number—with lying, this MAGAZINE 57 (1984), 105–108.
3. Thomas M. Thompson, *From Error-Correcting Codes through Sphere Packings to Simple Groups*, Carus Monograph #21, Mathematical Association of America, Washington, D.C., 1983, especially pp. 19–26.
4. Vera Pless, *Introduction to the Theory of Error-Correcting Codes*, John Wiley, New York, 1982.
5. F. J. MacWilliams and N. J. A. Sloane, *The Theory of Error-Correcting Codes*, North-Holland Pub. Co., 1977.
6. D. J. Kleitman, A. R. Meyer, R. L. Rivest, J. Spencer, K. Winklmann, Coping with errors in binary search procedures, *J. Comput. System Sci.* 20 (1980), 396–404.
7. T. A. Brylawski, *The Mathematics of Watergate (An Analysis of a Two-Party Game)*, Univ. of N. Carolina, 1978.
8. Donald J. Newman, *A Problem Seminar*, Springer-Verlag, New York, 1980.
9. Andrzej Pelc, Solution of Ulam's problem on searching with a lie, *J. of Combin. Theory* Ser. *A*, 44 (1987) 129–42.

Proof Without Words: Mollweide's Equation

$$(a-b)\cos\frac{\gamma}{2} = c\sin\left(\frac{\alpha}{2}-\frac{\beta}{2}\right)$$



—H. ARTHUR DEKLEINE
California Polytechnic State University
San Luis Obispo, CA 93407

A Brief, Subjective History of Homology and Homotopy Theory in This Century

(Lecture by Peter Hilton, followed by question-and-answer session)

PETER HILTON
SUNY at Binghamton
Binghamton, NY 13901

I have recently been recalling that about twenty-five years ago, when I first came to settle in this country, I was invited to participate in the celebration of the opening of the Mathematics Building, Van Vleck Hall, at the University of Wisconsin. On that occasion I learned a new American word, namely "banquet," which has a totally different meaning in the United States from the meaning that it has in Britain. But more importantly, I must recall the immense respect I felt for some of the after-dinner speakers who were able to make the recounting of an event last much longer than the event itself. So I'm very conscious of the fact that in attempting to recount to you the history of algebraic topology in this century, I must not make the recounting of this history last longer than the history. In fact, I must telescope it very dramatically, one might almost say, abruptly. So I apologize in advance that much of the treatment will be necessarily very superficial. I would like to start off with the first epoch which is up to 1926. And here the inspiration for homology theory comes from the work of Poincaré.

Poincaré, during a period earlier than the one I'm thinking of, had already invented or discovered, according to your philosophy, the fundamental group. But he published a series of papers in which he was studying what we would call algebraic varieties, the configuration of points in higher dimensional Euclidean space given by polynomial equalities and inequalities; and he was looking again at what we might call vector fields and generalizations of vector fields on such varieties. He was led through this study to look at what we would now call the homology of these varieties. In particular, he saw the significance for the solution of such vector field problems of what were then called the Betti numbers, which determine essentially the number of holes the configuration had. As a simple example, let us take the torus (FIGURE 1), which is of course an algebraic variety and has two very conspicuous one-dimensional holes. These are cycles which do not bound anything in the torus. Formally, the torus is itself a two-dimensional hole and any given point constitutes a zero-dimensional hole. Problems about the solutions of differential equations on the torus are different from problems relating, for example, to the sphere because the sphere has a different one-dimensional Betti number from that of the torus.

Poincaré also realized that there was a further subtlety, that is to say, there was a phenomenon which today is described as the phenomenon of the torsion coefficients.

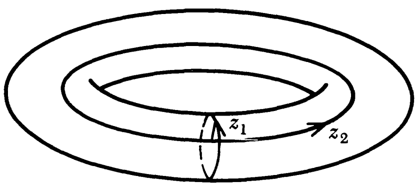


FIGURE 1
The torus, with its two basic one-dimensional cycles or holes, z_1 and z_2.

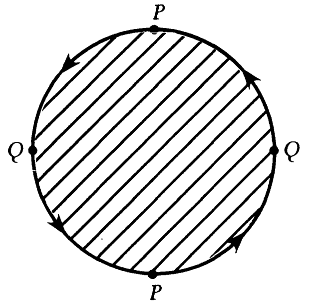


FIGURE 2
The real projective plane, with its basic one-dimensional cycle $z_1 = PQP$, such that z_1 does not bound but $2z_1$ bounds the disk.

Essentially, what one means by torsion coefficients can be demonstrated by the real projective plane, which I can represent by a circular disk, with diametrically opposite points identified on the boundary of the disk (FIGURE 2). It is not realizable in three-dimensional space, let alone two, so I make no apology for having to represent it this way on the blackboard. Now if we look at the path from P back to P, that is a cycle. That cycle does not in itself bound anything. But if you repeat the cycle then twice the cycle bounds the disk. This exemplifies the phenomenon of the existence of cycles which do not themselves bound, but multiples of them bound.

I want to stress at this point that we are dealing with numbers. These Betti numbers are numbers and in each dimension there are Betti numbers. The torsion coefficients are also numbers.

Other names that one should associate with this period of the great pioneers are J. W. Alexander (one usually associates the name of Alexander with knots but he did many other things that I will refer to), and Veblen (he wrote one of the great books, *Analysis Situs*, which was the early name for topology—the essential difference between the words analysis situs and topology is a choice between the Latin and Greek cultures; they essentially mean the same thing). The word topology, Veblen felt, had been pre-empted, because the word topology exists outside mathematics. Also, one should mention here the name of Brouwer, and I am most happy to do so because I understand Karel de Bouvère spoke here on Brouwer. In particular, with Brouwer one associates the idea of the degree of a map, which I shall be referring to later. I should also say that van Kampen and Lefschetz did pioneering work. These will come back into the story, even though I list them in this very early period. To go into any detail on the contribution of any one of them would certainly occupy the entire lecture. So what I really want to do is proceed to what I regard as one of the golden years, 1927. In regard to Lefschetz, however, I would like to make a remark of a personal nature, because it seems to me that, of all of the mathematicians, it is to Lefschetz one should give the credit for introducing systematically the notion of a polyhedron in a generalized sense. When I began a very enjoyable collaboration with Jean Pedersen, I discovered rather early on that we were separated by the misfortune of a common language, in which we used precisely the same terms but with different meaning. In particular, this word 'polyhedron' was almost responsible for the breaking up of a beautiful friendship. What does 'polyhedron' mean and what does it mean to classify polyhedra? To the topologist today, this is such a standard term, meaning the underlying topological space of a much more general type of combinatorial structure than that which is admitted by the geometer, the combinatorial geometer; and the classification of polyhedra, according to the topologist, is by homeomorphism generally, possibly by combinatorial equivalence, but again in a sense different from that used by the geometer (see FIGURE 3). I think we can say that our trouble stems from Lefschetz. It's a very key notion, this idea of homeomorphisms between polyhedra; it raised many questions, more questions than it answered, as any good mathematical notion will.

FIGURE 3
The cube and the octahedron, equivalent to the topologist but not to the geometer.

So now, with this very brief introduction to that early period, I want to discuss those years 1926/27. The reason I pick out these years is because these are the years when Alexandroff and Hopf were in Göttingen. Both of them were there as guests, a splendid example of the efficacy of having mathematical guests! Alexandroff was from the Soviet Union and Hopf, at that time, was from Berlin. They were both of them very much impressed with the work of Lefschetz; and, in particular, they were discussing the Lefschetz fixed point theorem. They recognized that this was in some way, which they began to put their fingers on, a generalization of the Euler–Poincaré characteristic. In connection with these Betti numbers which Poincaré had established, the so-called Euler characteristic was a topological invariant. The Lefschetz fixed point theorem, expressed in admittedly somewhat clumsy notation, they saw to be closely related. Indeed, the Lefschetz fixed point theorem, applied to the identity map from a space into itself, seemed to give the Euler–Poincaré characteristic. But, very significantly, there was also Emmy Noether in Göttingen; she would not have been there but for Hilbert's insistence. He felt that Göttingen was a place for mathematicians and not for sexism—which, in those days, was a very special point of view. Emmy Noether recognized that what Alexandroff and Hopf were talking about and what Lefschetz had talked about should not be thought of as numbers but should be thought of as Abelian groups. So really one should credit Emmy Noether, not with the discovery of these topological invariants but with understanding their mathematical place. Thus Emmy Noether recognized the homology *groups*, and that the Betti numbers and torsion coefficients were merely numerical invariants of isomorphism classes of finitely-generated Abelian groups. If you take a finitely-generated Abelian group A, then it can be written as the direct sum of a free Abelian group F and a family of cyclic groups A/h_i, where $h_1|h_2|\cdots$. Moreover the rank of the free Abelian part F and these numbers h_i are invariants of the group A. If A is the homology group in dimension d, say, then its rank is the dth Betti number. And the h_i are the torsion coefficients. Now you see that this is an enormous improvement over just a consideration of the numbers, because this immediately gives you the opportunity of adopting a far more dynamic approach to the whole question of homology; for then it is not simply that with a polyhedron you associate Betti numbers and torsion coefficients but with the polyhedron you associate homology groups. But, with Emmy Noether's improvement, there is very much more.

There is the natural question: how do you transform one polyhedron to another? How do you map one polyhedron into another? In the first place you have the idea of a simplicial map of the polyhedron and that induces a homomorphism of homology groups. I said, you remember, that the rank of F is an invariant—but F itself is not. A homomorphism from one Abelian group to another does not send the free part into the free part, nor does it preserve the nice little pieces, the Z/h_i. Things can get very much mixed up. The free part can go into the finite part. So you can only begin to understand the transformation of homology from this group-theoretical point of view. This was an enormous advance both conceptually and dynamically, in terms of the real understanding of what you have. It also posed the very obvious question: simplicial maps induce homomorphisms of homology groups. What can one say in general about merely continuous functions of the underlying polyhedra? Now I have not undertaken to talk about the complete history of the *Hauptvermutung*, which would again occupy a lot of time. So I must leave on one side such questions about the different combinatorial structures on the polyhedron, but what this throws into prominence immediately is the question: can you approximate to any continuous map of the underlying topological spaces by a nice combinatorial transformation of the polyhedra? The answer is yes; that is the simplicial approximation theorem, which dates from about this time. And so the picture begins to emerge of what homology is all about, and we agree first to look at this restricted class of spaces, the polyhedra and the simplicial maps between them, as a combinatorial structure. Here you can say you can do some forgetting, so that the polyhedra can be just thought of as topological spaces—we can forget their combinatorial structure—and the simplicial maps are then simply continuous functions. From the polyhedra and the simplicial maps you can construct, first, what are called chain complexes and chain maps. That means you look at the chains on the space. They are the linear combinations of the simplexes, the generalized triangles into which you have subdivided your space to make it a polyhedron. And finally you take the homology groups of the chain complex and get the homology groups of the space. You define first chain complexes and the chain maps, then homology groups and homomorphisms. For the polyhedron, you pick any polyhedral structure on the space, and you prove that the resulting homology groups only depend on the underlying space. That is the essential statement of the topological invariance of the homology groups. You go back from an arbitrary continuous function to a simplicial map by an approximation. You have choices of approximations but, whatever approximation you use, the homomorphism induced on homology groups is the same. This diagram here (FIGURE 4), where I have jumped many, many decades, shows essentially what these people were moving towards as they were elucidating the ideas of homology. Central to these ideas is the fact that the homology groups are really *homotopy* invariants, that is, homotopic continuous maps induce the same homology homomorphism. A nice example is furnished by maps f of the n-sphere S^n to itself. Now H_nS^n is cyclic infinite, so the induced homology homomor-

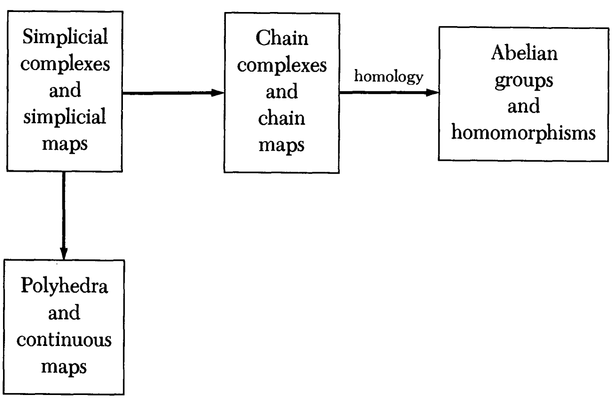


FIGURE 4
Scheme of the homology theory of polyhedra. Note that there are crucial notions of *homotopy* associated with continuous maps, simplicial maps and chain maps.

phism in dimension n is really just an integer, called the *degree* of f. The Brouwer–Hopf Theorem asserts that the degree is the unique invariant of the homotopy class of f.

I should also mention here the first person in this account who, as far as I know, is still alive: Vietoris, an Austrian mathematician, living in Innsbruck. Vietoris is one of the two who should be credited with seeing that you could define homology groups without the space having to be the underlying space of a polyhedron. With just the notion of a topological space, intrinsically defined, Vietoris defined homology groups (it even appears that Vietoris, independently of Emmy Noether, recognized the importance of the group concept). So there are homology groups of arbitrary spaces. He did it by means of coverings of the space by open sets. The other person we credit here is a Czech mathematician, Čech. They did it independently, as far as I can see, but they did it differently. They both did it by considering coverings of a space by open sets; the open sets in Čech's definition behaved like the vertices of a simplicial complex. According to Čech, then, a finite number of open sets span a simplex if and only if they have a non-empty intersection. Thus he saw these open sets as the vertices of a complex, so, from the present-day point of view, what Čech did was to study the nerve of a covering, which is an abstract simplicial complex which has a chain complex and homology groups attached.

Vietoris did something different. He took the open sets and said that a collection of points of the space constitute (in some sense) a simplex, if they all lie in one of the sets of the covering. These two points of view we now see as being, in a very real sense, dual to each other. The first person who made the duality of the Vietoris-Čech definitions precise was Hugh Dowker, a very fine Canadian mathematician, who, I regret to say, died recently. Also I should mention, in this connection, Mayer, who is known, and always will be known—as Heine is known as part of Heine-Borel—as part of Mayer-Vietoris. There is the Mayer-Vietoris sequence in homology. Hopf also credits Mayer, independently of Emmy Noether, with recognizing that groups were involved in the definition of homology, in a paper he published in 1929. But if I go on a bit from that time, I should mention that in 1932 there appeared a book by Alexandroff which was very influential and in 1935 there was the great book of Alexandroff and Hopf. There was a Volume I but there was never a Volume II. (The reason Volume II never appeared was the advent of cohomology.) This was an extremely influential book and was a sort of bible for the study of algebraic topology. It was a very beautifully written work. The original was, of course, in German but even if you do not understand German, it was easier to understand than most mathematical books written in English. The purpose of the treatment was to make the subject crystal clear. The pictures were beautiful.

Many things happened in 1935, which was very much a golden year. There was an international meeting in Moscow, in the summer. Hopf sent to Moscow his young student, Stiefel, who had begun a study of the existence of solutions of differential equations from the homological point of view and had come up with a certain idea which we now call *characteristic classes*. In Moscow at the meeting Stiefel read his paper, and in the audience was Hassler Whitney. (There is another living mathematician!) Hassler Whitney came up to Stiefel after and said: "This is remarkable, it is almost exactly what I have been doing"—in his study of (what we would now call) fiber spaces. Alexandroff, too, said that these fiber spaces are what they (the Russian school) had been doing. There they were called twisted products. So there was an extraordinary confluence of ideas revealed at this Moscow meeting. We now talk of the Stiefel-Whitney classes. These are thought of as characteristic cohomology classes on a real differentiable manifold. They are completely understood now but were then in their infancy. Also, this was the year when cohomology emerged and it emerged with the understanding that in cohomology you have a ring structure. Now cohomology was very slow to emerge for the simple reason that Emmy Noether's point of view was simply not understood by topologists. Despite the obvious advantages of the algebraic viewpoint, topologists continued to think exclusively geometrically. And I put the emphasis on the word 'exclusively'. That is what was wrong. And it is a remarkable fact about Hopf, the greatest of them all, that he could never feel comfortable with the idea of cohomology. The reason goes back to the idea of chains to which I have already (inadequately) referred. A chain is a linear combination of simplexes. For example, if you have some sort of triangulation, then you think of a chain as involving, in the one-dimensional case, the edges with certain multiplicities. Hopf always thought of it as some sort of path that was traced out on the polyhedron. And the current attitude towards homology and cohomology was this: if I, for example, look at a particular edge, this oriented edge, the two vertices of the edge, one with coefficient $+1$ and one with coefficient -1, that is, with this edge I will associate its boundary. The other thing I can do is associate, with this edge, all the triangles of which it is a side, that is, its coboundary (FIGURE 5). So in one case you lower the dimension and in the other case you raise it. The lowering of the dimension is homology, the raising of the dimension is co-homology. That was the current point of view and, indeed, under Alexandroff's influence, for many, many years the Russians continued to talk about lower and upper homology. But the point of view is flawed because, in the sense of linear algebra, cohomology is dual to homology, that is to say, if we think of 1-chains as linear combinations of edges, we should think of 1-cochains as functions of those edges. So we should distinguish between an edge and the function that takes on the value 1 on the edge. It is exactly the difference between a basis element of a vector space and the associated basis element of its dual. So cochains have to be thought of as functions on the simplexes. And although Hopf recognized that point of view he could not adopt it because to him homology was all about geometry. The idea that you were looking at functions which take values in an arbitrary Abelian group and which are defined on the edges or triangles of a polyhedron was a point of view that was totally uncongenial to him.

Cohomology was a very long time emerging because it was incorrectly regarded. There was also the feeling that it should be possible to introduce a multiplicative structure into cohomology which is not present in homology. Many attempts were made to do this. Alexander, whom I have mentioned, was one of the pioneers here and the attempt was finally successful in 1935. Of course, it enriches enormously the structure and adds to the discrimination that you get through homology theory, because you could very well have two topological spaces such that the Abelian group structure in cohomology is the same but in fact they have different multiplicative structures. If you take the torus, and on the other hand the following configuration (FIGURE 6), then the homology of the latter has the same additive structure as that of the torus, but it is distinguished from the torus by the multiplicative structure of its

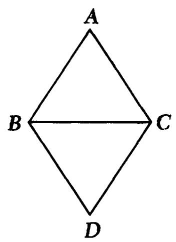


FIGURE 5
The boundary of BC is $C-B$. The coboundary of BC is $ABC+DBC$.

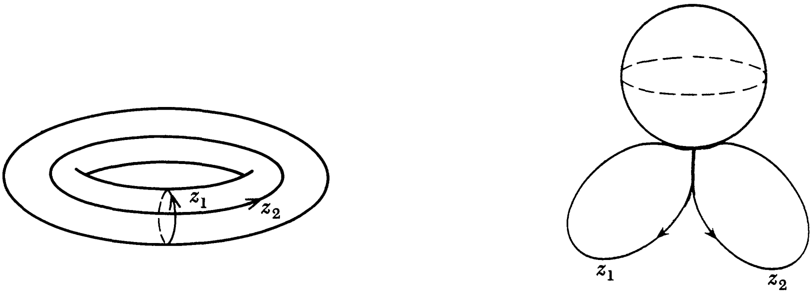


FIGURE 6

The torus and the "fake torus" consisting of two circles and a sphere joined at a point. The product z_1z_2 on the torus gives the torus itself, but $z_1z_2 = 0$ on the fake.

cohomology. The two one-dimensional cocycles on the torus multiply together to give the torus but in the other configuration they multiply together to give zero. So this was the great refinement of cohomology theory. It's interesting in this connection that Hopf, for whom I have enormous respect, didn't feel comfortable with this. Hopf had earlier seen, in the case of manifolds, that if you have a mapping f of manifolds, an m-dimensional manifold M mapped to an n-dimensional manifold N, then you have intersection rings of the manifolds. An intersection ring means that you take two cycles and look at their intersection in general position on the manifold. Hopf proceeded from there to define something that he called the *Umkehrshomomorphismus*, the backward homomorphism. The backward homomorphism was something that was induced by f which would go from the homology of N in dimension $(n-p)$ to the homology of M in dimension $(m-p)$ for arbitrary p. He defined it very carefully and he gave it this name the *Umkehrshomomorphismus*. What is it? Well, under the duality present in a manifold, the $(n-p)$ homology of N is essentially the same as the pth cohomology of N, as N is n-dimensional, and the $(m-p)$th homology of M is the same as the pth cohomology of M. So this homomorphism is simply the induced map in cohomology theory. Hopf defined this backward homomorphism and drew attention to the strange way that it changed the dimension in homology and went in the wrong direction. Of course it goes in the wrong direction because cohomology is contravariant, being based on dual vector spaces. You have a linear map from one vector space to another and the dual map maps the dual vectors back in the other direction. Hopf, with his wonderful, wonderful insight, got the right idea but stopped short of clarifying it in that way. But having made the only criticism one could possibly make of Hopf's mathematics let us move on to one of his tremendous contributions, also in 1935. In December of 1935 there was a meeting in Geneva and at this meeting Élie Cartan drew attention to the remarkable property of the classical Lie groups. He pointed out that all the classical Lie groups had the property that their Betti numbers—he was still talking about Betti numbers—were just like those of products of odd-dimensional spheres. So if you took any one of the classical Lie groups, that is the series of orthogonal groups, the unitary groups, and the symplectic groups, each one of those groups behaved, in a way, like a Cartesian product of odd-dimensional spheres. And this had simply been a matter of, you might say, empirical observation; that is to say, Cartan knew them all and their Betti numbers, and he just looked at them all and it was true for them all. But there was no explanation. He challenged people to produce an explanation. These facts, by the way, go back to Brouwer, Pontrjagin, and Ehresmann. And then he asked the next natural question, you take the five exceptional Lie groups, is it also true for them?

Why do the classical Lie groups always look like products of odd-dimensional spheres? Hopf thought about this and he came up with the answer. In his answer he pointed out that you needed very little of the structure of the classical Lie groups. Essentially all you needed is to say that you have a topological space together with a continuous multiplication on that space which has a two-sided identity. So we assume, of course, a compact space so that our homology will be finitely generated and a multiplication with a two-sided identity. That is all! This has always struck me as being a piece of incredible genius, because at the time that Hopf was working there were known exactly two examples of this phenomenon which were not Lie groups. One was the seven-dimensional sphere and it is surely somewhat unexciting to be told that the seven-dimensional sphere behaves like a product of odd-dimensional spheres! And the other is the real projective seven-dimensional space. It is almost equally unexciting because the real projective seven-dimensional space is an orientable manifold of which the seven-sphere is a two-sheeted covering. Obviously from the point of view of Betti numbers it behaves exactly like the seven-dimensional sphere. So there was no interesting example—but Hopf gave this as the explanation and this was the birth of the whole theory of Hopf algebras which is now a tremendous industry. Today we have infinitely many examples of the so-called Hopf manifolds which are not even topological groups. So as you see much happened in 1935.

It is now time to talk of the homotopy groups. There is a beautiful trade-off between the homology groups and the homotopy groups. The homology groups are terribly difficult to define but once you have defined them they are very easy to calculate. The homotopy groups are terribly easy to define but essentially impossible to calculate. The homotopy groups generalize the fundamental group. For the fundamental group you look at just the homotopy classes of maps of a circle into your space and, for the higher homotopy groups, you map spheres.

Actually, Hurewicz should not be credited with the actual invention of the homotopy groups. Really the credit for the invention should go to Čech. At a meeting in Vienna in 1931 Čech gave a paper in which he described certain groups from the homotopy point of view. He had no applications of these groups. Moreover, he had only one theorem, that they were commutative. And he was persuaded by people, and we know that Alexandroff played a role here, that they could not be interesting, because it was thought that any information that could be obtained from Abelian groups must come from the homology. Hurewicz redefined the homotopy groups and immediately gave important applications in a series of four notes which were intended as preliminary publications. In that series of four papers he showed the significance of what we now call obstruction theory. Essentially, as Steenrod was later to remark and codify, the basic problem you are facing in topology can so often be represented in the following way. You have a configuration X and you have a configuration Y, you have a subspace of X called L, and a mapping g from L into Y. The question is—can that continuous function g be extended to X? It's amazing how many questions inside and outside topology can be reduced to that. What Hurewicz showed was that this type of question could be answered in terms of certain obstructions which are cohomology classes of X modulo L with coefficients in the homotopy groups of Y. (In fact you usually cannot answer the question because you cannot calculate the obstructions!) The whole of obstruction theory was made absolutely systematic by Eilenberg. Hurewicz showed the significance of the homotopy groups and there is one great theorem that provides the link between homotopy and homology called the Hurewicz Isomorphism. Hurewicz pointed out that there was always a homomorphism going from the nth homotopy group to the nth homology group. If the space is such that the first $n-1$ homotopy groups vanish ($n \geqslant 2$), then this is an isomorphism. Thus the first place where the Hurewicz homomorphism is interesting it is an isomorphism. This generalizes the classical result already known essentially to Poincaré, the case $n=1$ where the first homology group is the fundamental group Abelianized. For the higher dimensions the homotopy groups are already Abelian so you don't have to Abelianize them. And that is the best that can be said of Alexandroff's point of view; that is, the first place where the homotopy groups come into play they are just homology groups. After that, their divergence is very significant, so you can say that homology and homotopy are proceeding together essentially as complementary concepts. And if I could just make one remark about what I regard as my own small contribution to this evolution, it would be my work with Eckmann. We showed that, though the homology and homotopy groups are essentially different, the method of construction of cohomology groups and of homotopy groups can be regarded as dual manifestations of exactly the same process. That is to say, the actual structure of cohomology theory can be mirrored by the structure of homotopy theory. Of course the results we get provide a vital link between the two theories.

One name I have not mentioned is Henry Whitehead. So let me say that what Henry Whitehead did involved a very beautiful idea. To go back to the beginning of homology, a topological space admitting homology was originally endowed with a combinatorial structure. Vietoris and Čech freed it of this combinatorial structure by defining homology on an arbitrary topological space. Homotopy theory was originally defined for an arbitrary topological space and what Henry Whitehead did was to impose a combinatorial structure on the space and show how this combinatorial structure on the space could, in fact, lead you to insights into its homotopy groups. These results first appeared in papers he wrote before the war and he then rewrote them afterwards. Whitehead said about his prewar work that he shared with Karl Marx the property of being frequently quoted and never read. And after the war he tried to deal with this by recasting a lot of his work in algebraic language. I was his first student after the Second World War; so I came under his influence in that period and that probably accounts a great deal for my early taste.

Question: When I studied algebraic topology I never realized that Emmy Noether had anything to do with it. Is this commonly known?

Answer: No. Hopf was very clear about this, about her tremendous contribution, this wonderful insight that she had. He said that he would never have realized that, in doing homology, they were talking about Abelian groups until she pointed this out. He saw the significance of the algebraic viewpoint in homology, but his difficulty was with cohomology. Hopf said that there was this wonderful atmosphere in Göttingen when they all got together and talked. Emmy Noether listened and then she came back and said, "Well, what you're really talking about is Abelian groups." They had these Betti numbers and they told her that the Betti numbers were for manifolds, but she said that they were talking about Abelian groups. And then she said that once you're talking about groups you must be talking about homomorphisms, but in the earlier proofs of invariance, you don't find any use of that fact. There is a good book by Andrew Wallace on homology theory. He gives, in a tiny little lemma tucked away, the fact that a map between spaces induces a homomorphism of their homology groups and that the composite of two maps induces the composite of the homomorphisms. Well, that's what topological invariance is all about, and the lemma makes it plain that you're transferring one theory into another, you're going from topology to algebra. That explains what you are doing in homology, and that is the point that Emmy Noether really clarified. Let me just tell one little story since this is Pólya country. I was present in Zürich on the occasion of Pólya's eightieth birthday and was invited to his birthday party. He and Hopf were discussing Emmy Noether and one of the two of them insisted that Emmy Noether was very ugly and the other hotly denied it.

Question: Why did some topologists fail to adopt the algebraic methods? Did they just feel that it would not yield worthwhile results?
Answer: It was the feeling that all of topology came from geometry and to some extent from analysis. Poincaré was, in the broad sense, an analyst. I think the idea of algebra as a tool in the hands of a geometer was a peculiar and strange idea. Now it seems much more natural but, in those days, it was very strange. In 1935, such a great mathematician as Élie Cartan was still just thinking in terms of Betti numbers. So this point of view was so foreign to them, this was what really held them up. Consider, for example, this business with cohomology and the contortions that many mathematicians went into with the 'upper boundary', with things always going wrong! They could see that though you got a map of chains from, say, a simplicial map, it didn't commute with the upper boundary, and Hopf had the idea that you had to think of a map going back the other way in some way. How? They just did not think in algebraic terms of cochains as functions on chains. As a further example, although it was clear to Alexandroff and to Hopf that it was natural, in connection with the Lefschetz fixed point theorem, to think in terms of homology with rational coefficients, they thought that homology with rational coefficients was just a means of getting rid of the torsion. They didn't think of it as a way of getting a vector space structure which is the way we would think of it now. Take rational coefficients, you get a nice rational vector space. So we can just talk about the dimension of the vector space. They just did not have these ideas. And I should say, too, that Henry Whitehead, in a paper before the War, introduced the key idea of an exact sequence in homotopy, but the way he wrote it out was extraordinarily obscure to us. Instead of just writing that, in the sequence, the kernel of a homomorphism is the same as the image of the preceding homomorphism, he has complicated statements because he distinguishes between the homotopy of maps into X and the homotopy of maps into a subspace A. In each case he wrote down what exactness meant, and so had two inclusions, one going one way, one going the other.

All of these algebraic ideas were a very long time in the making because the people doing homology and homotopy theory were not algebraists and the algebraists didn't take any interest. The only pure algebraist who took any interest was Emmy Noether.

Question: Now it's very common that you use the techniques of one branch of mathematics to solve problems in another. Was algebraic topology the first place where this happened?
Answer: Yes, in a systematic way, unless you would say that it was already being done in analytic number theory. There you are using methods of classical analysis in order to get results in number theory. There is a sense there though that you are not doing anything but the analysis. In topology, it's a sort of wedding of the two methods *simultaneously*—it's not the abandonment of one for the other but it's the establishing of links between the two, and in a sense it's gone so much further now because from topology one is led into the construction of new algebraic structures—and you can get algebraic results from the topology as well. By the use of covering spaces you can get theorems from combinatorial group theory that are terribly hard without the topological methods. So we have a two-way application. Cohomology theory has now spread over the whole of mathematics through differential equations, differential operators and so forth. And in algebraic geometry, of course, homology theory has become a basic tool.

NOTES

Tangents to Graphs

HUGH THURSTON
University of British Columbia
Vancouver V6T 1Y4

Introduction

Is it true that the graph of a function f has a nonvertical tangent at the point with coordinates $(c, f(c))$ if and only if $f'(c)$ exists? The answer is *no*: if the derivative exists the tangent exists, but not vice versa. However, if f is continuous at c, then the tangent exists if and only if the derivative exists. We shall prove these results in theorems 1 and 2 below.

Definition of tangent

Informally the tangent at a point P_0 to a curve C is the line through P_0 whose direction is the limit of the direction of the secant P_0P as P approaches P_0 along C. We cannot interpret "P approaches P_0 along C" as "the arc-length PP_0 approaches zero" unless we restrict ourselves to rectifiable curves. We therefore interpret it as "P is on C and close to P_0". We do not want to go into the tricky question of precisely what a curve is, so we allow our definition to apply to any set of points.

Definition 1. If C is a curve (or any set of points) and P_0 a nonisolated point of C, a line L through P_0 is a tangent to C at P_0 if for each positive number ε there is a positive number δ such that the angle between P_0P and L is smaller than ε for every point P of C within a distance δ of P_0.

It is clear that C cannot have more than one tangent at P_0, and that if C lies in a plane all its tangents lie in that plane.

The definition applies quite widely. Euclid would never have considered the set of all points on a given circle a rational distance from a given straight line to be a geometrical figure, but we can do so if we want to, and this ghost-circle has a tangent at each point according to the definition.

Informally, we might wonder whether a figure-of-eight curve which crosses itself at a point Q has two tangents or no tangents there; formally, our definition tells us there is no tangent there. This does not prevent a point moving round the figure-of-eight from having a well-defined direction of motion at Q. In fact, if it travels smoothly all the way round the figure-of-eight it will have two directions of motion at Q (at two different times).

This last example brings up the topic of parametrically-defined curves. Let C be the set of points with coordinates

$$(f(t), g(t))$$

for all t in the domains of f and g, and let P_0 be the point with parameter c. Is the line through P_0 with slope

$$\lim_{t \to c} \frac{g(t) - g(c)}{f(t) - f(c)}$$

a tangent to C at P_0? The answer is: *not necessarily*. If f and g are discontinuous we may be surprised. (Well, I was.) For example, let

$$f(t) = \begin{cases} t^{-1} & \text{if } t \neq 0 \\ 0 & \text{if } t = 0, \end{cases}$$

$$g(t) = \begin{cases} \sin(t^{-1}) & \text{if } t \neq 0 \\ 0 & \text{if } t = 0 \end{cases}$$

and $c = 0$. Then C is the set of points with coordinates $(u, \sin u)$ for every u, P_0 is the origin, and

$$\lim_{t \to c} \frac{g(t) - g(c)}{f(t) - f(c)} = \lim_{t \to 0} \frac{\sin(t^{-1})}{t^{-1}} = 0.$$

But the line through P_0 with slope zero is not a tangent to C: the tangent has slope 1. In fact, parametrically defined tangents are nonintuitive unless the parametric functions are continuous. The reason for the unexpected result is that the slope of the tangent is the limit of the slope of the secant as $(f(t), g(t))$ approaches $(f(c), g(c))$, and if f and g are not continuous this is not the same as the limit as t approaches c. I will hazard the opinion that all tangents defined via derivatives are nonintuitive unless the functions involved are continuous.

Differential geometers usually restrict themselves to the case where not only are f and g continuous but the function $t \to (f(t), g(t))$ from R to R^2 is one-to-one and has a continuous inverse, i.e. (f, g) is a homeomorphism. We can make the following definition.

Definition 2. If C is the set of points with coordinates $(f(t), g(t))$, where (f, g) is a homeomorphism, and if $f'(c)$ and $g'(c)$ exist and are not both zero, the line through the point $(f(c), g(c))$ in the direction of the vector $f'(c)\vec{i} + g'(c)\vec{j}$ is a *parametric tangent* to C at $(f(c), g(c))$.

It is easy to prove that every parametric tangent is a tangent under definition 1 ([**1**], page 1101), and it has been known since 1927 that there are curves which are defined via homeomorphisms and which have tangents that cannot be obtained as parametric tangents [**2**].

Conclusion

Let us now prove the results promised in the introduction. Let $f(x) = \sqrt{(1 - x^2)}$ if x is rational and $|x| \leq 1$, let $f(x) = -\sqrt{(1 - x^2)}$ if x is irrational and $|x| \leq 1$. The graph of f is a "ghost circle" and has a tangent at the point $(0, 1)$; however, f is not differentiable at 0, so we have the promised example of a tangent at $(c, f(c))$ with f nondifferentiable at c. This is another case where the function is discontinuous and the connection between tangent and derivative is not intuitive.

Finally, we turn to the promised theorems. In these theorems f is a function from R to R, c is a limit-point of its domain, and, for each h for which $c + h$ is in the domain of f, P_h is the point with coordinates $(c + h, f(c + h))$. In particular, P_0 is the point $(c, f(c))$.

Theorem 1. *If f is differentiable at c, the line L through P_0 with slope $f'(c)$ is a tangent to the graph of f at P_0.*

Proof. P_0 is not an isolated point of the graph because f is continuous at c and c is a limit point of the domain of f. For each positive ε there is a positive η such that lines with slopes between $f'(c)-\eta$ and $f'(c)+\eta$ make angles less than ε with L. There is then a positive δ such that

$$\left|\frac{f(c+h)-f(c)}{h}-f'(c)\right|<\eta \tag{i}$$

whenever $0<|h|<\delta$ and $c+h$ is in the domain of f. Whenever $0<|P_0P_h|<\delta$, $c+h$ is in the domain of f and $0<|h|<\delta$ and so (i) holds, the slope of P_0P_h lies between $f'(c)\pm\eta$, and P_0P_h makes an angle less than ε with L.

Theorem 2. *If f is continuous at c and the graph of f has a nonvertical tangent at P_0, then f is differentiable at c.*

Proof. Let ε be any positive number. Let m be the slope of the tangent, and let θ be the smaller of the following two angles: the angle between a line of slope m and a line of slope $m+\varepsilon$ and the angle between a line of slope $m-\varepsilon$ and a line of slope m. Let θ_h be the angle between P_0P_h and the tangent. If $\theta_h<\theta$, then

$$\left|\frac{f(c+h)-f(c)}{h}-m\right|<\varepsilon. \tag{ii}$$

There is a positive δ_1 such that $\theta_h<\theta$ whenever $|P_0P_h|<\delta_1$. There is a positive δ_2 such that $|f(c+h)-f(c)|<\frac{1}{2}\delta_1$ whenever $|h|<\delta_2$ and $c+h$ is in the domain of f.

Let δ be the smaller of $\frac{1}{2}\delta_1$ and δ_2. Then whenever $0<|h|<\delta$ and $c+h$ is in the the domain of f we have successively

$$\begin{gathered}|f(c+h)-f(c)|<\tfrac{1}{2}\delta_1,\\ 0<[f(c+h)-f(c)]^2+h^2<\delta_1^2,\\ 0<|P_0P_h|<\delta_1\\ \theta_h<\theta;\end{gathered}$$

and (ii) follows. Moreover c is a limit-point of the domain of f, otherwise P_0 would be an isolated point of the graph of f. Therefore, $f'(c)$ exists.

REFERENCES

1. H. A. Thurston, On the definition of tangent-line, *Amer. Math. Monthly* 71 (1964), 1099–1103.
2. G. Valiron, Sur les courbes qui admettent une tangente en chaque point, *Nouvelles Annales de Mathématique* 6 (1927), 46–51.

Proof without Words: The Characteristic Polynomials of AB and BA Are Equal

$$-\lambda^n|AB-\lambda I|=\left|\begin{pmatrix}A & AB-\lambda I\\ \lambda I & 0\end{pmatrix}\right|=\left|\begin{pmatrix}A & I\\ \lambda I & B\end{pmatrix}\begin{pmatrix}I & B\\ 0 & -\lambda I\end{pmatrix}\right|=\begin{vmatrix}A & I\\ \lambda I & B\end{vmatrix}(-\lambda)^n$$

$$-\lambda^n|BA-\lambda I|=\left|\begin{pmatrix}0 & \lambda I\\ BA-\lambda I & \lambda B\end{pmatrix}\right|=\left|\begin{pmatrix}A & I\\ \lambda I & B\end{pmatrix}\begin{pmatrix}-I & 0\\ A & \lambda I\end{pmatrix}\right|=\begin{vmatrix}A & I\\ \lambda I & B\end{vmatrix}(-\lambda)^n$$

—Sidney H. Kung
Jacksonville University
Jacksonville, FL 32211

THEOREM 1. *If f is differentiable at c, the line L through P_0 with slope $f'(c)$ is a tangent to the graph of f at P_0.*

Proof. P_0 is not an isolated point of the graph because f is continuous at c and c is a limit point of the domain of f. For each positive ε there is a positive η such that lines with slopes between $f'(c)-\eta$ and $f'(c)+\eta$ make angles less than ε with L. There is then a positive δ such that

$$\left|\frac{f(c+h)-f(c)}{h}-f'(c)\right|<\eta \tag{i}$$

whenever $0<|h|<\delta$ and $c+h$ is in the domain of f. Whenever $0<|P_0P_h|<\delta$, $c+h$ is in the domain of f and $0<|h|<\delta$ and so (i) holds, the slope of P_0P_h lies between $f'(c)\pm\eta$, and P_0P_h makes an angle less than ε with L.

THEOREM 2. *If f is continuous at c and the graph of f has a nonvertical tangent at P_0, then f is differentiable at c.*

Proof. Let ε be any positive number. Let m be the slope of the tangent, and let θ be the smaller of the following two angles: the angle between a line of slope m and a line of slope $m+\varepsilon$ and the angle between a line of slope $m-\varepsilon$ and a line of slope m. Let θ_h be the angle between P_0P_h and the tangent. If $\theta_h<\theta$, then

$$\left|\frac{f(c+h)-f(c)}{h}-m\right|<\varepsilon. \tag{ii}$$

There is a positive δ_1 such that $\theta_h<\theta$ whenever $|P_0P_h|<\delta_1$. There is a positive δ_2 such that $|f(c+h)-f(c)|<\frac{1}{2}\delta_1$ whenever $|h|<\delta_2$ and $c+h$ is in the domain of f.

Let δ be the smaller of $\frac{1}{2}\delta_1$ and δ_2. Then whenever $0<|h|<\delta$ and $c+h$ is in the domain of f we have successively

$$\begin{gathered}|f(c+h)-f(c)|<\tfrac{1}{2}\delta_1,\\ 0<[f(c+h)-f(c)]^2+h^2<\delta_1^2,\\ 0<|P_0P_h|<\delta_1\\ \theta_h<\theta;\end{gathered}$$

and (ii) follows. Moreover c is a limit-point of the domain of f, otherwise P_0 would be an isolated point of the graph of f. Therefore, $f'(c)$ exists.

REFERENCES

1. H. A. Thurston, On the definition of tangent-line, *Amer. Math. Monthly* 71 (1964), 1099–1103.
2. G. Valiron, Sur les courbes qui admettent une tangente en chaque point, *Nouvelles Annales de Mathématique* 6 (1927), 46–51.

Proof without Words: The Characteristic Polynomials of AB and BA Are Equal

$$-\lambda^n|AB-\lambda I|=\left|\begin{pmatrix}A & AB-\lambda I\\ \lambda I & 0\end{pmatrix}\right|=\left|\begin{pmatrix}A & I\\ \lambda I & B\end{pmatrix}\begin{pmatrix}I & B\\ 0 & -\lambda I\end{pmatrix}\right|=\begin{vmatrix}A & I\\ \lambda I & B\end{vmatrix}(-\lambda)^n$$

$$-\lambda^n|BA-\lambda I|=\left|\begin{pmatrix}0 & \lambda I\\ BA-\lambda I & \lambda B\end{pmatrix}\right|=\left|\begin{pmatrix}A & I\\ \lambda I & B\end{pmatrix}\begin{pmatrix}-I & 0\\ A & \lambda I\end{pmatrix}\right|=\begin{vmatrix}A & I\\ \lambda I & B\end{vmatrix}(-\lambda)^n$$

—SIDNEY H. KUNG
Jacksonville University
Jacksonville, FL 32211

A Difference Equation and Operation Counts in the Computation of Determinants

STEPHEN H. FRIEDBERG
Illinois State University
Normal, IL 61761

In many linear algebra texts, determinants of matrices are computed by means of cofactor expansion. These matrices are generally chosen to be of sizes smaller than 5×5. It is seldom emphasized that even with the aid of a large computer and for matrices of relatively small size, cofactor expansion is not practical. In this paper we shall solve a difference or recurrence equation to determine the number of multiplications required to evaluate a determinant by cofactor expansion. Most of this material assumes nothing beyond high school algebra.

Let $f(n)$ be the number of multiplications necessary to evaluate the determinant of an arbitrary $n \times n$ matrix A using cofactor expansion. Let C_{ij} denote the determinant of the $(n-1) \times (n-1)$ matrix obtained from A by deleting the ith row and jth column. Then, using the method of cofactor expansion along the first row, we have

$$\det A = a_{11}C_{11} - a_{12}C_{12} + \cdots + (-1)^{n+1}a_{1n}C_{1n},$$

where $\det A$ denotes the determinant of A and a_{ij} is the ijth entry of A (see [**1**], for more information regarding determinants). For example, if $n = 2$ and

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

then $\det A = aC_{11} - bC_{12} = ad - bc$. We see that 2 multiplications are necessary and so, $f(2) = 2$.

For $n \geqq 3$ the number of multiplications needed to evaluate each C_{1j} is equal to $f(n-1)$ by definition. Therefore, to compute $a_{1j}C_{1j}$, we need $f(n-1) + 1$ multiplications. Because there are n such terms above, we conclude that there are $n[f(n-1) + 1]$ multiplications needed to compute $\det A$. Hence, $f(n)$ satisfies

$$f(n) = n[f(n-1) + 1] \quad \text{for } n \geqq 3, \tag{1}$$

a *first-order linear (nonhomogeneous) difference equation with nonconstant coefficients* (see [**2**]). Our goal is to express f as a simple function of n, that is, we want to *solve* (1). Before we attempt this, we can quickly obtain a lower bound for $f(n)$ by noting that $f(n) = nf(n-1) + n > nf(n-1)$. So,

$$f(n) > nf(n-1) > n(n-1)f(n-2) > \cdots > n(n-1)(n-2)\cdots 3f(2) = n!.$$

This inequality shows that it will take more than 20! or approximately 2.4×10^{18} multiplications to evaluate the determinant of a 20×20 matrix. A computer performing one million multiplications per second will take over 77,000 *years* to determine $\det A$ by cofactor expansion! Later we will obtain a more precise estimate.

To solve the difference equation (1), we introduce the function $g(n) = f(n)/n!$. Since $f(2) = 2$, we have $g(2) = 1$. Substituting g into (1), we obtain

$$g(n)n! = n[g(n-1)(n-1)! + 1] = g(n-1)n! + n.$$

This inequality shows that it will take more than 20! or approximately 2.4×10^{18} multiplications to evaluate the determinant of a 20×20 matrix. A computer performing one million multiplications per second will take over 77,000 *years* to determine det A by cofactor expansion! Later we will obtain a more precise estimate.

To solve the difference equation (1), we introduce the function $g(n) = f(n)/n!$. Since $f(2) = 2$, we have $g(2) = 1$. Substituting g into (1), we obtain

$$g(n)n! = n[g(n-1)(n-1)! + 1] = g(n-1)n! + n.$$

So, $[g(n) - g(n-1)]n! = n$. Or, $g(n) - g(n-1) = 1/(n-1)!$. Therefore, $g(n) - g(2) = [g(n) - g(n-1)] + [g(n-1) - g(n-2)] + \cdots + [g(3) - g(2)] = 1/(n-1)! + 1/(n-2)! + \cdots + 1/2!$.
Thus, $g(n) = 1 + 1/2! + 1/3! + \cdots + 1/(n-1)!$. That is,

$$\begin{aligned} f(n) &= n![1 + 1/2! + 1/3! + \cdots + 1/(n-1)!] \\ &= n![1 + 1/2! + 1/3! + \cdots + 1/n!] - 1. \end{aligned}$$

Using the Maclaurin expansion of e^x evaluated at $x = 1$ with remainder, we obtain

$$\begin{aligned} f(n) &= n![e - 1 - e^c/(n+1)!] - 1 \\ &= n!(e-1) - e^c/(n+1) - 1, \end{aligned}$$

where $0 < c < 1$. Because $0 < e^c/(n+1) < 1$ for $n \geqq 2$, it follows that $f(n) =$ the largest integer less than or equal to $n!(e-1) - 1$.

Below is a table which lists the approximate times it takes a computer (performing one million multiplications per second) to apply cofactor expansion to an $n \times n$ matrix. The results are based only on the number of multiplications.

Order of the matrix	Time needed to compute the determinant by cofactor expansion
5	2.1×10^{-4} seconds
10	6.2 seconds
11	68.6 seconds
12	13.7 minutes
13	3.0 hours
14	41.6 hours
15	26.0 days
16	416.1 days
17	19.4 years
18	348.8 years
19	6,628.0 years
20	132,560.0 years

Because of the fast growth of $n!$, it is clear that cofactor expansion should be taught mainly as a theoretical tool. For example, cofactor expansion is particularly useful in defining a determinant inductively, in showing the existence of a determinant when defined as an n-linear function, in proving Cramer's rule, in applying the classical adjoint of a matrix, and in deducing properties of determinants by induction. A more practical method of computing a determinant is to use elementary row operations to reduce the matrix to triangular form. The determinant will then be the product of the diagonal entries. This method requires roughly $n^3/3$ multiplications, considerably less than our result. For example, using this method, it would take the computer above considerably less than one second to compute the determinant of a 20×20 matrix.

The number of additions and subtractions needed to evaluate a determinant by cofactor expansion may be handled similarly. For example, if $h(n)$ equals the number of such operations, then $h(n+1) = (n+1)h(n) + n$. From this difference equation, it follows that $h(n) = n! - 1$.

REFERENCES

1. S. Friedberg, A. Insel, and L. Spence, *Linear Algebra*, Prentice-Hall, Inc., 1979, Chapter 4.
2. S. Goldberg, *Introduction to Difference Equations*, Dover Publications, 1986.

Palindromes: Density and Divisibility

HARVEY SCHMIDT, JR.
Lewis and Clark College
Portland, OR 97219

Palindromes are positive integers, like 131 or 2552, which are the same whether they are read forwards or backwards. Most of us are quite familiar with the fact that a palindrome with an even number of digits is divisible by 11. This property relies on the fact that if a positive integer N is written in the usual base-10 digit notation as $N = n_t n_{t-1} \cdots n_1 n_0$ ($t+1$ digits) and $\mathrm{Alt}(N)$ is defined by $\mathrm{Alt}(N) = n_0 - n_1 + \cdots + (-1)^t n_t$, then $\mathrm{Alt}(N) \equiv N (\mathrm{mod}\, 11)$. Thus, if N is a palindrome with an even number of digits, $\mathrm{Alt}(N) \equiv 0$ and 11 divides N. The case for palindromes with an odd number of digits, however, is not as clean, for some of these palindromes are divisible by 11 and some are not. Calling a palindrome with an odd number of digits a **divisible** palindrome if it is divisible by 11, our focus then becomes: What can be said about divisible palindromes? What is special about them and just how hard are they to find?

First we deal with the notion of **density**, by which we mean the percentage of divisible palindromes among all palindromes with a fixed odd number of digits. For the purposes of computing this density, we consider a 5-digit palindrome such as 73437 as a 7-digit palindrome 0734370 or a 9-digit palindrome 007343700, and so forth. Thus, when we refer to the total number of 5-digit palindromes, we include the ordinary one, three, and five-digit palindromes (i.e., where no leading or trailing zeroes are present). In general, when speaking of the $2n+1$-digit palindrome $P = p_0 p_1 \cdots p_{n-1} p_n p_{n-1} \cdots p_1 p_0$ ($n \geq 1$), we allow blocks of equal length of both leading and trailing zeroes, so that the set of $2n+1$-digit palindromes includes all ordinary palindromes with $2k+1$ digits for $0 \leq k \leq n$. With this convention it is easy to see that there are precisely 10^{n+1} palindromes with $2n+1$ digits.

Intuition suggests that roughly $1/11$ of all palindromes with an odd number of digits should be divisible palindromes simply because every eleventh integer is divisible by 11 and because, when listing all $2n+1$-digit integers in their natural reduce the matrix to triangular form. The determinant will then be the product of the diagonal entries. This method requires roughly $n^3/3$ multiplications, considerably less than our result. For example, using this method, it would take the computer above considerably less than one second to compute the determinant of a 20×20 matrix.

The number of additions and subtractions needed to evaluate a determinant by cofactor expansion may be handled similarly. For example, if $h(n)$ equals the number of such operations, then $h(n+1) = (n+1)h(n) + n$. From this difference equation, it follows that $h(n) = n! - 1$.

REFERENCES

1. S. Friedberg, A. Insel, and L. Spence, *Linear Algebra*, Prentice-Hall, Inc., 1979, Chapter 4.
2. S. Goldberg, *Introduction to Difference Equations*, Dover Publications, 1986.

Palindromes: Density and Divisibility

HARVEY SCHMIDT, JR.
Lewis and Clark College
Portland, OR 97219

Palindromes are positive integers, like 131 or 2552, which are the same whether they are read forwards or backwards. Most of us are quite familiar with the fact that a palindrome with an even number of digits is divisible by 11. This property relies on the fact that if a positive integer N is written in the usual base-10 digit notation as $N = n_t n_{t-1} \cdots n_1 n_0$ ($t+1$ digits) and Alt(N) is defined by $\text{Alt}(N) = n_0 - n_1 + \cdots + (-1)^t n_t$, then $\text{Alt}(N) \equiv N (\text{mod}\, 11)$. Thus, if N is a palindrome with an even number of digits, $\text{Alt}(N) \equiv 0$ and 11 divides N. The case for palindromes with an odd number of digits, however, is not as clean, for some of these palindromes are divisible by 11 and some are not. Calling a palindrome with an odd number of digits a **divisible** palindrome if it is divisible by 11, our focus then becomes: What can be said about divisible palindromes? What is special about them and just how hard are they to find?

First we deal with the notion of **density**, by which we mean the percentage of divisible palindromes among all palindromes with a fixed odd number of digits. For the purposes of computing this density, we consider a 5-digit palindrome such as 73437 as a 7-digit palindrome 0734370 or a 9-digit palindrome 007343700, and so forth. Thus, when we refer to the total number of 5-digit palindromes, we include the ordinary one, three, and five-digit palindromes (i.e., where no leading or trailing zeroes are present). In general, when speaking of the $2n+1$-digit palindrome $P = p_0 p_1 \cdots p_{n-1} p_n p_{n-1} \cdots p_1 p_0$ ($n \geq 1$), we allow blocks of equal length of both leading and trailing zeroes, so that the set of $2n+1$-digit palindromes includes all ordinary palindromes with $2k+1$ digits for $0 \leq k \leq n$. With this convention it is easy to see that there are precisely 10^{n+1} palindromes with $2n+1$ digits.

Intuition suggests that roughly $1/11$ of all palindromes with an odd number of digits should be divisible palindromes simply because every eleventh integer is divisible by 11 and because, when listing all $2n+1$-digit integers in their natural order, the palindromes among them are almost evenly spaced. For example, among all 3-digit integers, every tenth integer is a palindrome with nine exceptions, and in each of these exceptions there is one additional integer interposed.

Indeed, it is possible to make this precise, for if d_n represents the number of divisible palindromes with $2n+1$ digits, then we show that

THEOREM A. $d_n = (1/11)(10^{n+1} + (-1)^n)$.

Thus, if D_n is the set of divisible palindromes among the set of all palindromes with $2n+1$ digits, Theorem A allows us to draw the

COROLLARY. $\mathrm{Prob}(D_n) = d_n/10^{n+1} = (1/11)(1 + (-1)^n/10^{n+1})$; *consequently*

$$\lim_{n\to\infty} \mathrm{Prob}(D_n) = 1/11,$$

as intuition suggested.

To establish Theorem A, notice that

$$\begin{aligned} P = p_0p_1 \cdots p_{n-1}p_np_{n-1} \cdots p_1p_0 \in D_n \\ \Leftrightarrow \mathrm{Alt}(P) = 2\left(p_0 - p_1 + \cdots + (-1)^{n-1}p_{n-1}\right) + (-1)^n p_n \equiv 0 \pmod{11} \\ \Leftrightarrow (-1)^{n+1}p_n \equiv 2\left(p_0 - p_1 + \cdots + (-1)^{n-1}p_{n-1}\right) \pmod{11}. \end{aligned}$$

Thus, given digits $p_0, p_1, \ldots, p_{n-1}$, there is a unique solution p_n in $\{0,1,\ldots,9\}$ to this latter equation except when either

$$n \text{ is odd and } p_0 - p_1 + \cdots + (-1)^{n-1}p_{n-1} \equiv 5 \pmod{11},$$

or

$$n \text{ is even and } p_0 - p_1 + \cdots + (-1)^{n-1}p_{n-1} \equiv 6 \pmod{11}.$$

(In these latter cases, a solution for the digit p_n would require that p_n be congruent to 10, modulo 11, which is not possible.) This means that the number d_n of solutions to $\mathrm{Alt}(P) \equiv 0 \pmod{11}$ is 10^n minus the number of these exceptions (which depend on the parity of n).

The cases for the exceptions are similar; we handle the odd case first. The equation $x_0 - x_1 + \cdots + (-1)^r x_r \equiv 5 \pmod{11}$ has solutions with $x_j \in \{0,1,\ldots,9\}$ if and only if $x_1 - x_2 + \cdots + (-1)^{r-1}x_r \not\equiv 5 \pmod{11}$. Thus, if t_r is the number of solutions of the equation $x_0 - x_1 + \cdots + (-1)^r x_r \equiv 5 \pmod{11}$, then $t_r = 10^r - t_{r-1}$ for $r \geq 1$. Since $t_0 = 1$ and $t_1 = 9$, this recursion formula and a simple induction argument show that $t_r = (1/11)(10^{r+1} + (-1)^r)$ for all $r \geq 0$.

The case in which n is even is a little more awkward. Let s_r be the number of solutions to the equation

$$x_0 - x_1 + \cdots + (-1)^r x_r \equiv y \pmod{11}, \quad \text{with } x_j \in \{0,1,\ldots,9\},$$

when $y = 6$, and let f_r be the number of solutions when $y = 4$. Since

$$x_0 - x_1 + \cdots + (-1)^r x_r \equiv 6 \pmod{11} \Rightarrow x_1 - x_2 + \cdots + (-1)^{r-1}x_r \not\equiv 4 \pmod{11},$$

and the same is true with the roles of the 6 and 4 reversed, we have $s_r = 10^r - f_{r-1}$ and $f_r = 10^r - s_{r-1}$. Again, since $s_0 = f_0 = 1$, induction shows that $s_r = f_r$ for all $r \geq 0$ so that $s_r = 10^r - s_{r-1}$. As in the odd case, the closed expression $s_r =$

$(1/11)(10^{r+1}+(-1)^r)$ follows from this recursion formula by induction.

Applying the observations of the previous two paragraphs to the digits of P means that the number of exceptions, t_n if n is odd or s_n if n is even, is the same regardless of the parity of n so that

$$d_n = 10^n - t_{n-1} = t_n = (1/11)\left(10^{n+1} + (-1)^n\right)$$

for all $n \geq 1$, as claimed in Theorem A.

Now that we know how often we should expect to find divisible palindromes, just what might be so special about them? A simple example sheds some light on what to expect. If P is the 7-digit palindrome 1918191 then $P = 11Q$, where $Q = 174381$. We can display this as

$$\begin{array}{rl}
\mathbf{174381} & \\
\times\ \mathbf{11} & \\
\hline
01010 & \leftarrow \text{“carries”} \\
\mathbf{174381} & \\
+\,\mathbf{174381} & \\
\hline
\mathbf{1918191} &
\end{array}$$

Although the "carries" in multiplication by 11 are always either 0 or 1, this diagram suggests that, in addition, the symmetry of P may force a symmetry in the carries and that there may be a connection between symmetrically situated digits in Q and the number of nonzero carries.

The hint in this example that the carries reflect the symmetry of the digits of P is of some interest in its own right and provides the stimulus for our second main result. If P is a divisible palindrome and $P = 11Q$, then we exploit this symmetry to analyze the connection between P and Q. Because $P \equiv \text{Alt}(P) \pmod{11}$ and because 11 divides P, it follows that 11 also divides $\text{Alt}(P)$. All that remains is to determine the quotient $\text{Alt}(P)/11$, namely

THEOREM B. *If* $P = 11Q$ *then* $\text{Alt}(P) = 11\text{Alt}(Q)$.

To set the notation for the proof of Theorem B, let $P = 11Q$, where $P = p_0 p_1 \cdots p_{n-1} p_n p_{n-1} \cdots p_1 p_0$ has $2n+1$ digits and $Q = q_{2n-1} q_{2n-2} \cdots q_1 q_0$, possibly with $q_{2n-1} = 0$. Then the equation $P = 11Q$ and the division algorithm imply that

$$\begin{aligned}
q_0 &= p_0 \\
q_0 + q_1 &= 10c_1 + p_1, \\
q_{i-1} + q_i + c_{i-1} &= 10c_i + p_i, \qquad 2 \leq i \leq n, \\
q_{i-1} + q_i + c_{i-1} &= 10c_i + p_{2n-i}, \qquad n+1 \leq i \leq 2n-1,
\end{aligned} \tag{1}$$

and

$$q_{2n-1} + c_{2n-1} = p_0,$$

where the p_i's and q_j's are the digits of P and Q, respectively, and the c_k's, the carries, are either 0 or 1.

The first step, in fact the crucial step, is to show that the c_i's do indeed inherit symmetry from the digits of P and that this symmetry, in turn, is imparted to certain differences involving the q_j's. From (1) we have that $q_0 = p_0$ and $q_{2n-1} + c_{2n-1} = p_0$; thus $c_{2n-1} = q_0 - q_{2n-1}$. Appealing to (1) twice more, once with $i = 1$ and once with

$i = 2n-1$, we may write $q_0 + q_1 = 10c_1 + p_1$ and $q_{2n-2} + q_{2n-1} + c_{2n-2} = 10c_{2n-1} + p_1$, so that

$$10(c_1 - c_{2n-1}) = -c_{2n-2} + (q_0 - q_{2n-1}) + (q_1 - q_{2n-2}) \\ = -c_{2n-2} + c_{2n-1} + (q_1 - q_{2n-2}). \tag{2}$$

If $c \neq c_{2n-1}$, then $c_{2n-1} = 0$ implies that $10 = -c_{2n-2} + (q_1 - q_{2n-2})$, while $c_{2n-1} = 1$ implies that $-11 = -c_{2n-2} + (q_1 - q_{2n-2})$, neither of which is possible since $|q_1 - q_{2n-2}| \leq 9$. Therefore, $c_1 = c_{2n-1}$, whereupon (2) implies that $c_{2n-2} = c_{2n-1} + (q_1 - q_{2n-2}) = c_1 + (q_1 - q_{2n-2})$, or that $q_{2n-2} - q_1 = c_1 - c_{2n-2}$.

A general recursive argument, along the lines of the previous paragraph, allows us to show

$$c_{2n-1} = q_0 - q_{2n-1},$$
$$c_i - c_{2n-i-1} = q_{2n-i-1} - q_i, \tag{3}$$

and

$$c_i = c_{2n-i}, \quad \text{for} \quad i = 1, \dots, n-1.$$

The next step is to develop an alternate expression for Alt(P) which depends solely on the carries. Solving each of the equations in (1) for the p_i's and using the telescoping nature of the alternating sum for Alt(P), we can write

$$\begin{aligned}\text{Alt}(P) &= p_0 - p_1 + \cdots + (-1)^n p_n + \cdots + p_2 - p_1 + p_0 \\ &= q_0 - (q_0 + q_1 - 10c_1) + (q_1 + q_2 + c_1 - 10c_2) - (q_2 + q_3 + c_2 - 10c_3) \\ &\quad + \cdots + (q_{2n-3} + q_{2n-2} + c_{2n-3} - 10c_{2n-2}) \\ &\quad - (q_{2n-2} + q_{2n-3} + c_{2n-2} - 10c_{2n-1}) + (q_{2n-1} + c_{2n-1}) \\ &= 11c_1 - 11c_2 + \cdots - 11c_{2n-2} + 11c_{2n-1};\end{aligned}$$

that is,

$$\text{Alt}(P) = 11(c_1 - c_2 + \cdots - c_{2n-2} + c_{2n-1}). \tag{4}$$

A proof of Theorem B is now at hand. Combining (3) and (4) we obtain

$$\begin{aligned}\text{Alt}(P) &= 11(c_1 - c_2 + \cdots - c_{2n-2} + c_{2n-1}) \\ &= 11[(c_1 - c_{2n-2}) - (c_2 - c_{2n-3}) + \cdots + (-1)^n(c_{n-1} - c_n) + c_{2n-1}] \\ &= 11[(q_{2n-2} - q_1) - (q_{2n-3} - q_2) + \cdots + (-1)^n(q_n - q_{n-1}) + (q_0 - q_{2n-1}) \\ &= 11(q_0 - q_1 + q_2 - \cdots + q_{2n-2} - q_{2n-1}) \\ &= 11\,\text{Alt}(Q).\end{aligned}$$

Concluding Remarks. First, simple examples show that the property Alt(P) = 11 Alt(Q) fails to distinguish divisible palindromes from palindromes with an even number of digits, e.g., try 4224 and 1331. Second, if P and Q are any two positive integers with the property that $P = 11Q$ and if the q_j's are defined as in (1), then (4) still holds. Moreover, if P is a palindrome with an even number of digits, then the c_k's are again symmetric as in (3). Although Alt(P) = 0 in this case, it is not possible to obtain a tidy relationship between the differences of the c_k's and the differences of the q_j's, so that an expression for Alt(Q) in terms of the c_k's alone is not possible. Nevertheless, it is quite straightforward to obtain an expression for Alt(Q) in terms of both the c_k's and the p_i's. Finally, all of the results in this note can be generalized to an arbitrary base b with 11 replaced by $b + 1$.

How to Beat Your Kids at Their Own Game

KENNETH M. LEVASSEUR
University of Lowell
Lowell, MA 01852

1. Introduction. The first game that my children learned, as two-year olds, was "Red or Black," in which they tried to guess the color of the top card in a deck. The top card is removed from the deck after each guess and the game ends when the deck is empty. Your score is the number of correct guesses that you make. We assume that a child's strategy is that he or she randomly guesses red or black for each card, not taking into account the colors of cards that have been removed from the deck. A child would then have an expected score of 26 for a standard deck. The most elementary form of card counting can be used to increase the expected score. If there are more cards remaining of one color, then that color should be guessed next. A question that is immediately posed is: Using card counting, what is the expected score for a standard 52 card deck? Or more generally: Using card counting, what is the expected score, $S(n)$, for a deck of n red and n black cards?

2. Game paths, expected score, and diagonal crossings. Each game of "Red or Black" can be viewed as a decreasing path through lattice points starting at (n, n) and ending at $(0,0)$ (see FIGURE 1). At each point in the path, the first coordinate represents the number of red cards left in the deck and the second coordinate represents the number of black cards. Each step in the path corresponds to the removal of one card from the deck; therefore exactly one of the coordinates decreases by one in each step. Any such path will be called a **game path.**

A paradoxical aspect to counting cards in "Red or Black" is that the expected score grows proportionally with the number of times that card counting gives no information. Those situations occur on the diagonal: red = black. Suppose that we started at (n, n) and that the first card in the deck is black. We can assume that this first card contributes $1/2$ to our expected score. Now suppose that our game path meets the

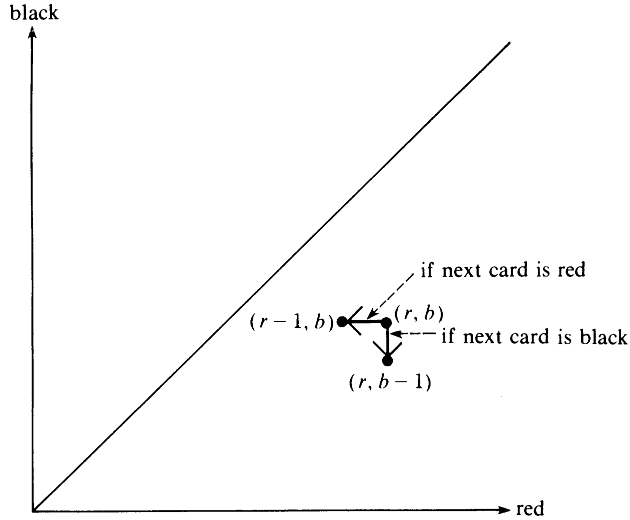


FIGURE 1

diagonal next at $(n-k, n-k)$. In the course of traversing this path we would continue to guess that the next card is red since red > black in that portion of the path; therefore, we will be correct exactly k times and incorrect $k-1$ times during this part of the game. In the path from (n, n) to $(n-k, n-k)$, our expected score is then $k+0.5$ of a possible $2k$. A child's strategy of random guessing would yield an expected score of k between (n, n) and $(n-k, n-k)$. Clearly, the more we visit the diagonal, the more frequently we will gain this extra 0.5 in our expected score.

Example. The game path in FIGURE 2 is randomly chosen from the $C(52,26)$ ways in which twenty-six red and twenty-six black cards can be arranged in a deck of fifty-two. This game path "visits" the diagonal 5 times, not counting $(0,0)$. At each of these visits, we assume that our guess is made at random. Therefore, our expected score for this game is 28.5.

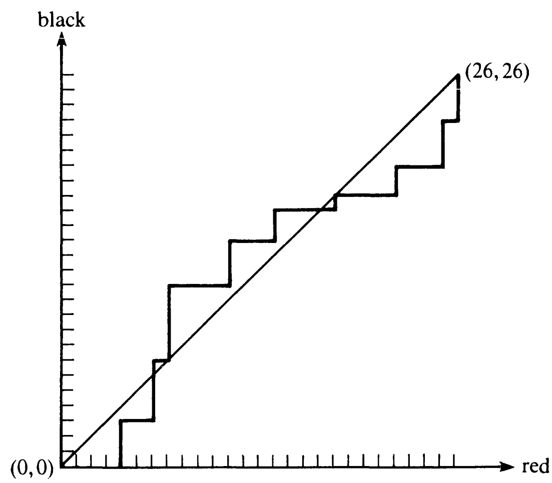


FIGURE 2

Let P be the set of all game paths. Since a game path is determined by the n positions of the red cards in a $2n$ card deck, there are $C(2n, n)$ different game paths. Let p be any game path; and let $v(p)$ be the number of diagonal visits by p. Given the observation about diagonal visits, the expected score given p is $n+0.5(v(p)-1)$. We consider visiting $(0,0)$ a diagonal visit; hence the -1 in this formula. Since we assume that the deck of cards is randomly shuffled, each of the $C(2n, n)$ paths is equally likely; thus

$$S(n) = \sum_{p \in P} (n + 0.5(v(p) - 1))/C(2n, n)$$

$$= n + 0.5\left(\left(\sum_{p \in P} v(p)/C(2n, n)\right) - 1\right).$$

This reduces our problem to the calculation of the average number of visits to the diagonal; or equivalently, to the calculation of the total number of diagonal visits in all game paths.

3. The total number of diagonal visits. We will use a combination of combinatorial averaging (as in [4]) and generating function techniques (as in [1]) to compute $V(n)$, the total number of diagonal visits by all game paths in P. Let

$$\chi(p,m)=\begin{cases}1 & \text{if } p \text{ visits } (m,m)\\ 0 & \text{otherwise.}\end{cases}$$

$$V(n)=\sum_{p\in P}\left[\sum_{m=0}^{n}\chi(p,m)\right]=\sum_{m=0}^{n}\left[\sum_{p\in P}\chi(p,m)\right]$$

$$=\sum_{m=0}^{n}\text{the number of paths that visit }(m,m)$$

$$=\sum_{m=0}^{n}C(2m,m)C(2(n-m),n-m).$$

The final equality is derived as follows (see FIGURE 3). If p visits (m,m), then the first $2(n-m)$ cards in the deck consist of exactly $n-m$ red cards and $n-m$ black cards, which can be arranged in $C(2(n-m),n-m)$ ways. Furthermore, the last $2m$ cards contain m red and m black, which can be arranged $C(2m,m)$ ways. Hence the product $C(2m,m)C(2(n-m),n-m)$ is the total number of game paths that visit (m,m).

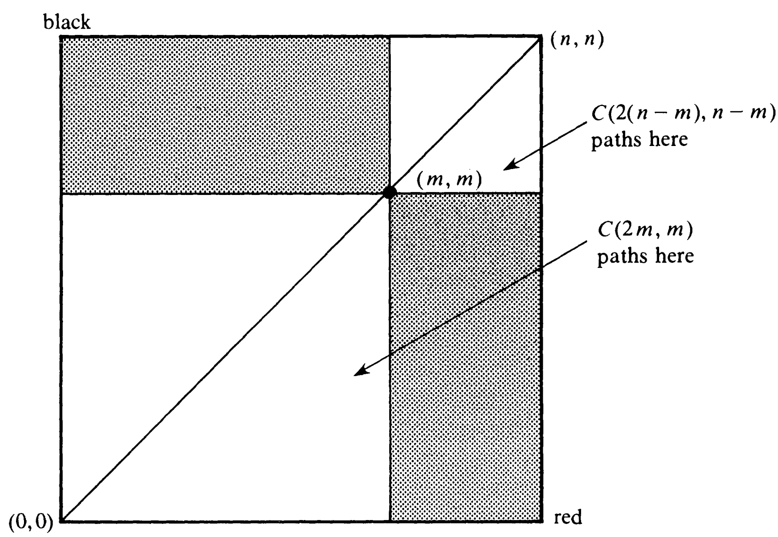


FIGURE 3

At this point, we remind the reader of some basic properties of sequences and generating functions. If A is a sequence of numbers, then the generating function of A is $G(A;z)=A(0)+A(1)z+A(2)z^2+A(3)z^3+\cdots$. For example, if $A(n)=ar^n$, $G(A;z)$ is a geometric series, which has closed form $a/(1-rz)$. The convolution of sequences A and B is a sequence $A*B$, where

$$(A*B)(n)=\sum_{m=0}^{n}A(m)B(n-m).$$

One of the most useful facts about generating functions is that $G(A*B;z)=G(A;z)G(B;z)$.

THEOREM. *Let V be the total number of diagonal visits by all game paths in P. Then $G(V;z)=1/(1-4z)$, which implies that $V(n)=4^n$.*

Proof. The key observation to be made is that the Taylor series expansion of $1/(\sqrt{1-4z})$ is

$$\sum_{n=0}^{\infty} C(2n, n)z^n.$$

Let $D(n) = C(2n, n)$. Our final expression for $V(n)$ above is $(D * D)(n)$; i.e., $V = D * D$. Therefore,

$$G(V; z) = G(D * D; z) = G(D; z)^2 = 1/(1-4z).$$

The only sequence having this generating function is the geometric sequence 4^n; therefore, $V(n) = 4^n$.

4. Return to the original problem, asymptotic estimates. With our calculation of $V(n)$, we have

$$S(n) = n + 0.5((4^n/C(2n, n)) - 1). \tag{1}$$

The first term of this expression represents the portion of the expected score that we would expect from making random guesses, while the second term represents the advantage from counting cards. For values of n such as 26 this second term is difficult to calculate; hence we turn to an asymptotic estimate.

Recall that if f, g, and h are sequences of real numbers, then $f(n) = g(n) + O(h(n))$ means that there exist integer N and real number C such that $|f(n) - g(n)| \leqslant C|h(n)|$ for all $n \geqslant N$. If $\lim_{n\to\infty} h(n) = 0$, then $g(n)$ can be used as an approximation of $f(n)$. Of course, the usefulness of the approximation depends on N and C.

Our estimate of $S(n)$ is based on Stirling's approximation of n-factorial,

$$n! = \sqrt{2\pi n}\,(n/e)^n + O\left(\sqrt{2\pi/n}\,(n/e)^n\right).$$

This formula can be substituted into (1) to obtain

$$S(n) = n + 0.5(\sqrt{\pi n} - 1) + O(1/\sqrt{n}). \tag{2}$$

Since $\lim_{n\to\infty} 1/\sqrt{n} = 0$, we can approximate $S(n)$ with the first two terms in (2).

5. Results of a computer simulation. To get a feel for how good our approximation of $S(n)$ is, a simulation of "Red or Black" was run for $n = 26$ (the usual deck size) and for $n = 100$. The results of the simulation appear in TABLE 1. They seem to indicate that the $O(1/\sqrt{n})$ term in (2) is small enough to estimate $S(n)$ to the nearest integer, at least.

TABLE 1. Results of a Computer Simulation

n	Games	Average Score	Approximation of $S(n)$
26	300	30.007	30.019
100	200	108.290	108.362

6. Acknowledgments and comment. Thanks go to my mother for teaching "Red or Black" to my children and to D. Knuth, whose study of the toilet paper problem in [2] led me to the idea of game paths.

Several persons have commented that with the simple form of $V(n)$, a simpler derivation might be possible; but I know of none.

REFERENCES

1. A. W. Doerr and K. M. Levasseur, *Applied Discrete Structures for Computer Science*, SRA, Chicago, 1985.
2. D. E. Knuth, The toilet paper problem, *Amer. Math. Monthly* 91 (1984), 465–470.
3. P. W. Purdom and C. A. Brown, *The Analysis of Algorithms*, Holt, Rinehart and Winston, New York, 1985.
4. Herbert S. Wilf, Some examples of combinatorial averaging, *Amer. Math. Monthly* 92 (1985), 250–261.

When Does the Symmetry Property Hold?

GLORIA OLIVE
University of Otago
Dunedin, New Zealand

If a and b are integers, the usual combinatorial argument can be used to show that the symmetry property for the binomial coefficients,

$$\binom{a+b}{a}=\binom{a+b}{b}, \tag{A}$$

holds when

$$a\geq 0 \quad \text{and} \quad b\geq 0. \tag{B}$$

Various students and others have used (A) more freely—and without justification. In order to determine necessary conditions for (A) to hold, we will first let x represent a complex number and n represent a positive integer. Then the adoption of the usual definition

$$\binom{x}{0}\equiv 1 \quad \text{and} \quad \binom{x}{n}\equiv\frac{x(x-1)\cdots(x-n+1)}{n!} \tag{C}$$

and the usual convention

$$\binom{x}{-n}\equiv 0 \tag{D}$$

help to reveal that (A) may not hold. For example,

$$\binom{-n}{-n}\neq\binom{-n}{0}$$

when n is a positive integer.

We have found that the following problem can lead students and others to the answer of the question, "When does (A) hold?"

Several persons have commented that with the simple form of $V(n)$, a simpler derivation might be possible; but I know of none.

REFERENCES

1. A. W. Doerr and K. M. Levasseur, *Applied Discrete Structures for Computer Science*, SRA, Chicago, 1985.
2. D. E. Knuth, The toilet paper problem, *Amer. Math. Monthly* 91 (1984), 465–470.
3. P. W. Purdom and C. A. Brown, *The Analysis of Algorithms*, Holt, Rinehart and Winston, New York, 1985.
4. Herbert S. Wilf, Some examples of combinatorial averaging, *Amer. Math. Monthly* 92 (1985), 250–261.

When Does the Symmetry Property Hold?

GLORIA OLIVE
University of Otago
Dunedin, New Zealand

If a and b are integers, the usual combinatorial argument can be used to show that the symmetry property for the binomial coefficients,

$$\binom{a+b}{a}=\binom{a+b}{b}, \tag{A}$$

holds when

$$a\geq 0 \quad \text{and} \quad b\geq 0. \tag{B}$$

Various students and others have used (A) more freely—and without justification. In order to determine necessary conditions for (A) to hold, we will first let x represent a complex number and n represent a positive integer. Then the adoption of the usual definition

$$\binom{x}{0}\equiv 1 \quad \text{and} \quad \binom{x}{n}\equiv\frac{x(x-1)\cdots(x-n+1)}{n!} \tag{C}$$

and the usual convention

$$\binom{x}{-n}\equiv 0 \tag{D}$$

help to reveal that (A) may not hold. For example,

$$\binom{-n}{-n}\neq\binom{-n}{0}$$

when n is a positive integer.

We have found that the following problem can lead students and others to the answer of the question, "When does (A) hold?"

Problem. If a and b are integers, determine whether (A) holds "always," "sometimes," or "never," when

(1) $0 \leq a \leq a+b$,
(2) $a+b<a<0$,
(3) $0 \leq a+b \leq a$,
(4) $a \leq a+b<0$,
(5) $a \leq 0 \leq a+b$,
(6) $a+b<0 \leq a$,
(7) $a+b=0$,
(8) $ab=0$,
(9) $[a+b \geq 0]$ or $[(a<0)$ and $(b<0)]$.

We note that the answers are "never" in (4) and (6), "sometimes" in (8), and "always" in the other 6 cases. Also, (9) provides the desired answer to our question. Hence, using the symbols of logic, we have

$$\left[\binom{a+b}{a}=\binom{a+b}{b}\right] \leftrightarrow [(a+b \geq 0) \vee ([a<0] \wedge [b<0])] \tag{E}$$

and, therefore,

$$\left[\binom{g}{h}=\binom{g}{g-h}\right] \leftrightarrow [(g \geq 0) \vee (g<h<0)] \tag{F}$$

when a, b, g, and h are integers.

In order to prove (E) we will let

$$p:(a+b \geq 0), \quad q:(a<0), \quad \text{and} \quad r:(b<0).$$

Then a direct argument [using both (C) and (D)] yields

$$[p \vee (q \wedge r)] \rightarrow (\mathrm{A}).$$

In order to establish the converse,

$$(\mathrm{A}) \rightarrow [p \vee (q \wedge r)],$$

we will prove the equivalent (contrapositive) implication,

$$[(\sim p) \wedge ([\sim q] \vee [\sim r])] \rightarrow [\sim (\mathrm{A})]$$

(using the Laws of De Morgan), which can also be stated as

$$[([\sim p] \wedge [\sim q]) \vee ([\sim p] \wedge [\sim r])] \rightarrow [\sim (\mathrm{A})]$$

(using a distributive law). Now let us assume we have $[\sim p] \wedge [\sim q]$. Then $(a+b<0)$ and $(a \geq 0)$, and hence $(b<0)$. Therefore,

$$\binom{a+b}{a} \neq 0 \quad \text{and} \quad \binom{a+b}{b}=0,$$

and we thus have $\sim$ (A). A similar argument (or "symmetry") can be used to show that $\sim$ (A) also follows from $[\sim p] \wedge [\sim r]$, and the proof is now complete.

In addition to Result (E), the above proof has provided us with an application of the basic laws of logic. A further application of the Laws of De Morgan can take us from Result (E) to

$$\left[\binom{a+b}{a} \neq \binom{a+b}{b}\right] \leftrightarrow [(a+b<0) \wedge (ab \leq 0)]. \tag{E'}$$

Various inequalities can now be obtained. For example, if c and k are nonnegative integers, Result (E′) can be used to establish results such as the following.

$$\left[\binom{k-1}{c-1} \neq \binom{k-1}{k-c}\right] \leftrightarrow [c \geq k = 0]. \tag{G}$$

$$\left[\binom{k+c-1}{c-1} \neq \binom{k+c-1}{k}\right] \leftrightarrow [c = k = 0]. \tag{H}$$

It was the binomial coefficients that appear in (G) and (H) together with some observations by one of my students (Stephen Cranefield) that led me to investigate the symmetry property. For in a search for "binomial functions" [i.e., functions of c and k that satisfy the functional equation

$$B_{c+1}^{k} = \sum_{j=0}^{k} B_c^j B_1^{k-j} \tag{I}$$

(when c and k are nonnegative integers)] I found that

$$B_c^k = \binom{k-1}{k-c} \quad \text{and} \quad B_c^k = \binom{k+c-1}{k} \tag{J}$$

qualify, and then assumed that the symmetry property could be used to show that

$$B_c^k = \binom{k-1}{c-1} \quad \text{and} \quad B_c^k = \binom{k+c-1}{c-1} \tag{K}$$

also qualify. But they do not qualify—as can be seen by letting $c = 0$ in (I). [It can be shown that (I) holds for the 2nd function of (K) if and only if $c \geq 1$, and for the 1st function of (K) if and only if $c = k = 2n$ when n is a positive integer.]

The binomial coefficients in (K) also have some basic combinatorial interpretations. For example when c and k are positive integers, the number of solutions of

$$x_1 + \cdots + x_c = k \tag{L}$$

is

$$N(c, k; > 0) = \binom{k-1}{c-1}, \tag{M}$$

when each x_i is required to be a positive integer [for each solution can be obtained by placing $c-1$ markers into $k-1$ (linearly ordered) spaces], and

$$N(c, k; \geq 0) = \binom{k+c-1}{c-1} \tag{N}$$

when each x_i is required to be a nonnegative integer [for by letting $y_i = 1 + x_i$, this result can be obtained from (M) after each x_i is replaced by y_i]. A legitimate application of the symmetry property can now provide us with the alternative answers of

$$\binom{k-1}{k-c} \quad \text{and} \quad \binom{k+c-1}{k},$$

respectively.

We now observe that

$$N(c,k;>0)=0 \quad\text{and}\quad N(c,k;\geq 0)=\delta_{k0}$$

when k is either zero or a negative integer, and hence

$$N(c,k;>0)=\binom{k-1}{k-c} \quad\text{and}\quad N(c,k;\geq 0)=\binom{k+c-1}{k} \tag{O}$$

both hold when k is any integer and c is a positive integer—but (M) and (N) do not. Thus, the binomial coefficients in (O) can serve as "continuations" of the binomial coefficients in (M) and (N), respectively.

These "continuations" also arise in connection with formal series. For when c is a positive integer we can obtain

$$\left(\sum_{k\geq 1} t^k\right)^c=\sum_{k\geq c}\binom{k-1}{c-1}t^k=\sum_{k\geq c}\binom{k-1}{k-c}t^k \tag{P}$$

and

$$\left(\sum_{k\geq 0} t^k\right)^c=\sum_{k\geq 0}\binom{k+c-1}{c-1}t^k=\sum_{k\geq 0}\binom{k+c-1}{k}t^k; \tag{Q}$$

but when c is any integer, both (P) and (Q) need to be modified by deleting the series in the middle.

Hence, there are times when it is useful to have an answer to the question raised in the title of this article.

I want to dedicate this article to my Part 3 (honours) students at the University of Otago—especially Stephen Cranefield—who helped me to become aware that (A) cannot be freely used.

Extending the Converse of Pons Asinorum

JOHN P. HOYT
1800 Vollage Circle 147
Homestead Village
Lancaster, PA 17603

In the 1985 American High School Mathematics Examination (AHSME), sponsored in part by the Mathematical Association of America, problem 28 asked for side b of triangle ABC, given that angle C was three times angle A, and that sides a and c were 27 and 48, respectively. In the official Solutions Pamphlet [1], neither of the two given solutions made use of the relationship that exists between the sides (a, b, c) of triangle ABC when angle C is three times angle A:

$$ab^2=(c+a)(c-a)^2. \tag{1}$$

This interesting but not well-known relationship (I have not seen it in any book) is respectively.

We now observe that

$$N(c, k; > 0) = 0 \quad \text{and} \quad N(c, k; \geq 0) = \delta_{k0}$$

when k is either zero or a negative integer, and hence

$$N(c, k; > 0) = \binom{k-1}{k-c} \quad \text{and} \quad N(c, k; \geq 0) = \binom{k+c-1}{k} \tag{O}$$

both hold when k is any integer and c is a positive integer—but (M) and (N) do not. Thus, the binomial coefficients in (O) can serve as "continuations" of the binomial coefficients in (M) and (N), respectively.

These "continuations" also arise in connection with formal series. For when c is a positive integer we can obtain

$$\left(\sum_{k\geq 1} t^k\right)^c = \sum_{k\geq c} \binom{k-1}{c-1} t^k = \sum_{k\geq c} \binom{k-1}{k-c} t^k \tag{P}$$

and

$$\left(\sum_{k\geq 0} t^k\right)^c = \sum_{k\geq 0} \binom{k+c-1}{c-1} t^k = \sum_{k\geq 0} \binom{k+c-1}{k} t^k; \tag{Q}$$

but when c is any integer, both (P) and (Q) need to be modified by deleting the series in the middle.

Hence, there are times when it is useful to have an answer to the question raised in the title of this article.

I want to dedicate this article to my Part 3 (honours) students at the University of Otago—especially Stephen Cranefield—who helped me to become aware that (A) cannot be freely used.

Extending the Converse of Pons Asinorum

JOHN P. HOYT
1800 Vollage Circle 147
Homestead Village
Lancaster, PA 17603

In the 1985 American High School Mathematics Examination (AHSME), sponsored in part by the Mathematical Association of America, problem 28 asked for side b of triangle ABC, given that angle C was three times angle A, and that sides a and c were 27 and 48, respectively. In the official Solutions Pamphlet [1], neither of the two given solutions made use of the relationship that exists between the sides (a, b, c) of triangle ABC when angle C is three times angle A:

$$ab^2 = (c+a)(c-a)^2. \tag{1}$$

This interesting but not well-known relationship (I have not seen it in any book) is easily proved without using trigonometry, and it is one of a sequence of similar theorems [2] which can be proved in a manner both simple and unusual for Euclidean geometry. It is the purpose of this note to show one such method for proving these theorems (or discovering the relationships), to show how the numbers used on the 1985 AHSME can be obtained, and to show how any number of integral-sided primitive triangles can be found in which one angle is n times another, $n = 2, 3, 4 \dots$.

Let CBA_n be a triangle in which angle C is n times angle A_n (n being a positive integer) and let the sides of the triangle be c_n, b_n, and a_n. FIGURE 1 shows such a triangle (with $n > 1$) inscribed in a circle. Since angle $BCA_n >$ angle A_n, $BA_n > CB$. Hence we can construct $A_nA_{n-1} = CB$ and, drawing lines CA_{n-1} and BA_{n-1}, we have triangle CBA_{n-1} with angle $BCA_{n-1} = (n-1)$ times angle A_{n-1} of that triangle. Then the various chords can be labelled as shown and the following statements are apparent from Figure 1.

$$a_{n-1} = a_n \quad \text{(identity)} \tag{2}$$

$$b_{n-i} = c_n \quad \text{(chords of equal arcs)} \tag{3}$$

$$c_{n-1} = \left(c_n^2 - a_n^2\right)/b_n. \tag{4}$$

Statement (4) is obtained from Ptolemy's Theorem: the product of the diagonals of a cyclic quadrilateral is equal to the sum of the products of the opposite sides of the quadrilateral. We have used Ptolemy's Theorem with appropriate lettering from FIGURE 1 and then solved for c_{n-1}. Statements (2), (3), and (4) enable us to start with the relationship between the sides of triangle CBA_{n-1} and find the relationship between the sides of triangle CBA_n.

Now assume that $n = 2$ in FIGURE 1. Then triangle CBA_1 is isosceles (converse of Pons Asinorum) with

$$a_1 = c_1. \tag{5}$$

Substituting in (5) for a_1 from (2) and for c_1 from (4), we have

$$a_2 = \left(c_2^2 - a_2^2\right)/b_2 \quad \text{or} \quad a_2 b_2 = c_2^2 - a_2^2. \tag{6}$$

Thus (6) is a necessary condition for angle C of triangle CBA_2 to be two times angle A_2. Using the cosine law on triangle CBA_2 together with (6) will show that $\cos A_2 = b_1/2a_1 = \cos A_1$ as it should.

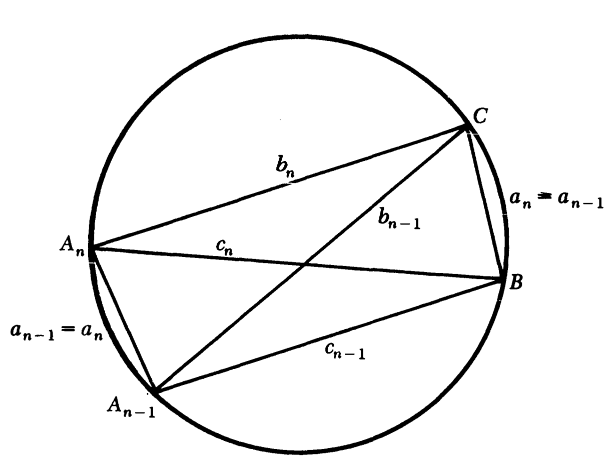


FIGURE 1

Now assume that $n=3$ in FIGURE 1. Then (6) gives the relationship between the sides of triangle CBA_2 and substituting the values given in (2), (3), and (4) in (6), we find (after simplifying) the relationship between the sides of triangle CBA_3 to be

$$a_3b_3^2=(c_3+a_3)(c_3-a_3)^2. \tag{7}$$

Thus (7) is a necessary condition for angle C of triangle CBA_3 to be three times angle A_3. Statement (7) without subscripts is (1).

Using the recurrence relations (2), (3), and (4), on (7) will give

$$a_4b_4^2(a_4+b_4)=(c_4^2-a_4^2)(c_4^2-a_4^2-a_4b_4), \tag{8}$$

which is a necessary condition for angle C of triangle CBA_4 to be four times angle A_4.

Likewise it can be shown that a necessary condition for angle C of triangle CBA_5 to be five times angle A_5 is

$$a_5b_5^4=(c_5-a_5)\Big(\big[c_5^2-a_5^2\big]^2-2a_5^2b_5^2-a_5b_5^2c_5\Big). \tag{9}$$

For any other positive integral value of n, similar but more involved relationships can be found.

Now we show how the numbers can be obtained for problem 28 on the 1985 AHSME. We start with an isosceles triangle CBA_1. To be sure that we can find a triangle for which $\angle C_3=3\angle A_1$, we must be sure that $\angle A_1<180°/4$. Since $\cos A_1=b_1/2a_1$, we must have $b_1/2a_1>\cos 45°$; i.e., $b_1/2a_1>1/\sqrt{2}$. Also $b_1<2a_1$ by the triangle inequality. Meeting these requirements are $a_1=c_1=3$ and $b_1=5$.

We now want to find the sides of triangle CBA_2. Formulas (2) and (3) can readily be used to go from triangle CBA_n to triangle CBA_{n-1} or vice versa. But to use formula (4) to go from triangle CBA_{n-1} to triangle CBA_n we need to make slight changes in it with the help of (2) and (3) to get

$$b_n=(b_{n-1}^2-a_{n-1}^2)/c_{n-1}. \tag{10}$$

Statement (10) is also apparent from FIGURE 1.

Using (2), (3), and (10) with $n=2$ gives $a_2=a_1=3$, $c_2=b_1=5$, and $b_2=(5^2-3^2)/3=16/3$. Since we want an integral-sided triangle, we multiply each of these values by 3 to get the sides of a triangle similar to triangle CBA_2 with $a_2=9$, $c_2=15$, and $b_2=16$. Using our recursion formulas (with $n=3$) on this new triangle gives $a_3=9$, $c_3=16$, and $b_3=35/3$. Once more we multiply by 3 to get $a=27$, $c=48$, and $b=35$, which were the sides of triangle ACB of problem 28. It can be verified that these numbers satisfy (1), which can be used to find b given a and c. It can also be verified by the cosine law that $\cos A=5/6=\cos A_1$.

The solution I would give to problem 28 if I did not know (6) or (7) would be based on FIGURE 2, which is the construction of FIGURE 1 used twice. Then reading from FIGURE 2 and using Ptolemy's theorem twice: from quadrilateral CA_2A_1B, $y^2=(27)(48)+27^2$ from which $y=45$ and from quadrilateral CA_3A_2B, $x=(48^2-27^2)/y=35$.

It should now be clear how one can easily find integral-sided triangles such that one angle is n times another for any positive integral value of n. The following examples illustrate some possible values of the sides of a triangle in which the angle opposite side c_n is n times the angle opposite side a_n. Notice in each example we proceed until the value of $b_n<a_n$ and after each step we multiply by a_1 to get integral sides. The values of a_1 and b_1 in each example are chosen so that we can successively arrive at

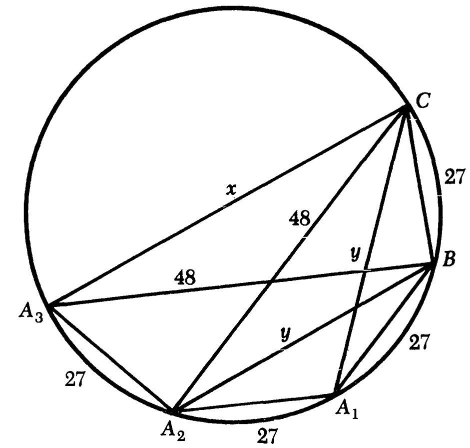


FIGURE 2

values for sides of a CBA_n triangle with $n = 2, 3, 4, 5, 6$.

Example 1.	$a_1 = 3$,	$a_2 = 9$.	*Example* 2.	$a_1 = 2$,	$a_2 = 4$,	$a_3 = 8$.
(to $n = 2$)	$c_1 = 3$,	$c_2 = 12$.	(to $n = 3$)	$c_1 = 2$,	$c_2 = 6$,	$c_3 = 10$.
	$b_1 = 4$,	$b_2 = 7$.		$b_1 = 3$,	$b_2 = 5$.	$b_3 = 3$.

Example 3.	$a_1 = 3$,	$a_2 = 9$,	$a_3 = 27$,	$a_4 = 81$.		
(to $n = 4$)	$c_1 = 3$,	$c_2 = 15$,	$c_3 = 48$,	$c_4 = 105$.		
	$b_1 = 5$,	$b_2 = 16$,	$b_3 = 35$.	$b_4 = 31$.		
Example 4.	$a_1 = 4$,	$a_2 = 16$,	$a_3 = 64$,	$a_4 = 256$,	$a_5 = 1024$.	
(to $n = 5$)	$c_1 = 4$,	$c_2 = 28$,	$c_3 = 132$,	$c_4 = 476$,	$c_5 = 1220$.	
	$b_1 = 7$,	$b_2 = 33$,	$b_3 = 119$,	$b_4 = 305$,	$b_5 = 231$.	
Example 5:	$a_1 = 6$,	$a_2 = 36$,	$a_3 = 216$,	$a_4 = 1296$,	$a_5 = 7776$,	$a_6 = 46656$.
(to $n = 6$)	$c_1 = 6$,	$c_2 = 66$,	$c_3 = 510$,	$c_4 = 3234$,	$c_5 = 17214$,	$c_6 = 72930$.
	$b_1 = 11$,	$b_2 = 85$,	$b_3 = 539$,	$b_4 = 1869$,	$b_5 = 12155$,	$b_6 = 30421$.

In general, if we start with an isosceles triangle CBA_1 with $a_1 = c_1 = a$ and $b_1 = b$, where a and b are positive integers with

$$2a\cos(180°/[n+1]) < b < 2a \text{ for any integer } n > 1, \tag{11}$$

we can find expressions for the sides of an integral-sided triangle in which angle C is n times angle A. We designate such a triangle by $C_nB_n'A_n$. It is possible to find expressions for the sides, a_n', c_n', and b_n', in terms of a, b, and n, but except for a_n' whose value is a^n, it is more practical to find expressions for the sides of triangle $C_nB_n'A_n$ in terms of a and b for a given n.

To find expressions for the sides of triangle $C_2B_2'A_2$ in terms of a and b, we follow the method of the preceding numerical problems using (2), (3), and (10) and then multiplying each side of the resulting CBA_2 triangle by a. Thus for triangle $C_2B_2'A_2$ we have $a_2' = a^2$, $c_2' = ab$, $b_2' = b^2 - a^2$. It can be seen that continuing in this manner will give $a_n' = a^n$ (as already stated) and $c_n' = ab'_{n-1}$. However, if we use (10) multiplied by $\underline{a}$ to find b_n', we are faced with divisions by polynomials of increasing length as n increases and we have no guarantee (at this point) that the divisions will be free of remainders. Therefore, we show a replacement formula for (10) in which division is avoided. To derive this replacement formula, consider FIGURE 3.

FIGURE 3 shows triangle CBA_n which has for its sides the sides of triangle $C_nB_n'A_n$ each divided by a^{n-1}. Thus $CB = a^n/a^{n-1} = a$ and $CA_n = b_n'/a^{n-1}$. Likewise, the figure shows two sides of triangle CBA_{n-1}, which is similar to triangle $C_{n-1}B_{n-1}'A_{n-1}$ with a ratio of similitude of $1/a^{n-2}$ and two sides of triangle CBA_{n-2}, which is

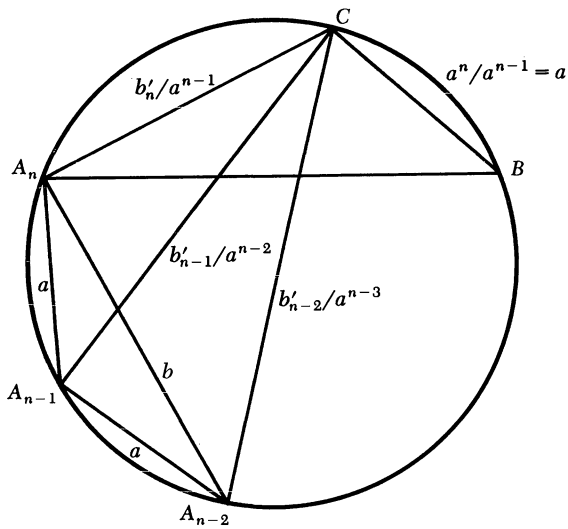


FIGURE 3

similar to triangle $C_{n-2}B'_{n-2}A_{n-2}$ with a ratio of similitude of $1/a^{n-3}$. This should explain the lettering of the figure. The construction is exactly the same as that of FIGURE 1 used twice. Using Ptolemy's theorem on quadrilateral $CA_nA_{n-1}A_{n-2}$ gives

$$ab'_n/a^{n-1} = bb'_{n-1}/a^{n-2} - ab'_{n-2}/a^{n-3}. \tag{12}$$

Multiplying both members of (12) by a^{n-2} gives

$$b'_n = bb'_{n-1} - a^2b'_{n-2}, \qquad n > 2. \tag{13}$$

Using (13) with initial conditions $b'_1 = b$ and $b'_2 = b^2 - a^2$ enables one to find easily b'_n for small values of n and together with the values of a'_n and c'_n already given one can quickly find the values given in (14), (15), and (16).

$$\text{For } C_3B'_3A_3,\ a'_3 = a^3,\quad c'_3 = a(b^2 - a^2),\qquad b'_3 = b^3 - 2a^2b. \tag{14}$$

$$\text{For } C_4B'_4A_4,\ a'_4 = a^4\quad c'_4 = a(b^3 - 2a^2b),\qquad b'_4 = b^4 - 3a^2b^2 + a^4. \tag{15}$$

$$\text{For } C_5B'_5A_5,\ a'_5 = a^5,\quad c'_5 = a(b^4 - 3a^2b^2 + a^4),\quad b'_5 = b^5 - 4a^2b^3 + 3a^4b. \tag{16}$$

The values given in (14), (15), and (16) (after dropping the primes) will satisfy equations (7), (8), and (9), respectively, and can be looked upon as two-parameter solutions to these Diophantine equations. The values of a and b used in (14), (15), and (16) do not have to be restricted by (11) to get values which satisfy (7), (8), and (9) but they must be so restricted if the resulting numbers are to satisfy the triangle inequality.

It is interesting to note that (13) is a second-order homogeneous linear difference equation and together with initial conditions $b'_1 = b$, $b'_2 = b^2 - a^2$, can be solved for b'_n in terms of a, b, and n. If this is done, it will be found that the roots of the auxiliary equations are complex numbers whose moduli are a and whose amplitudes are $\angle A_1$ in one case $-\angle A_1$ in the other, and that

$$b'_n = a^n(\text{cis}[n+1]A_1 - \text{cis} - [n+1]A_1)/2i \sin A_1,$$
$$n = 0, 1, 2, \ldots, \qquad \cos A_1 = b/2a, \tag{17}$$

which becomes

$$b_n' = a^n(\sin[n+1]A_1)/\sin A_1, \qquad n = 0,1,2,\ldots, \qquad \cos A_1 = b/2a \quad (18)$$

when simplified using $\operatorname{cis} x = \cos x + i \sin x$.

We have included 0 in the values of n since we let $b_0' = 1$, which is consistent with $b_1' = b$ and $b_2' = b^2 - a^2$ for initial conditions for finding a particular solution of (13), and which is easier to use than $b_2' = b^2 - a^2$. The value for b_n' given in (18) can be obtained immediately by using the sine law for triangles on triangle $C_nB_n'A_n$. See [3] for the connection with Chebyshev polynomials.

The impracticality of using (18) to find the values of b_n' given in (14), (15), and (16), and especially for larger values of n than 6, is now evident even if one uses

$$\sin nx = n \sin x \cos^{n-1}x - \binom{n}{3}\sin^3 x \cos^{n-3}x + \binom{n}{5}\sin^5 x \cos^{n-5}x - \cdots. \quad (19)$$

Acknowledgement. The author is grateful to Professor John Rigby, University College, Cardiff, Wales for his helpful suggestions (as referee) that were used in the preparation of this paper for publication.

REFERENCES

1. Back copies of AHSME examinations and solution pamphlets: Professor Walter E. Mientka, Dept. of Mathematics and Statistics, University of Nebraska, Lincoln, NE 68588.
2. W.W. Williams, A generalization of a property of the 4, 5, 6 triangle, *Mathematical Gazette* 60 (June 1976), 130–131.
3. Cornelius Lanczos, *Applied Analysis*, Prentice-Hall, 1956, pp. 178–179.

As for plays, he detested the theatre, and said he would as lief do a sum as follow a plot on the stage.

William Dean Howells,
My Mark Twain

There were people who were prophesying the collapse of European civilization on the grounds that there was no longer any faith, any love, any simplicity or any goodness left in mankind; and it is significant that these people were all bad at mathematics at school. This only went to convince them, later on, that mathematics, the mother of the exact natural sciences, the grandmother of engineering, was also the arch-mother of that spirit from which, in the end, poison-gases and fighter aircraft have been born.

Robert Musil, *The Man Without Qualities*. Translated by Eithne Wilkins and Ernst Kaiser, 1953; written before 1930.

PROBLEMS

LOREN C. LARSON, *editor*
St. Olaf College

BRUCE HANSON, *associate editor*
St. Olaf College

Proposals

To be considered for publication, solutions should be received by May 1, 1989.

1307. *Proposed by Zun Shan and Edward T. H. Wang, Wilfrid Laurier University, Canada.*

Let ΔABC be a triangle with altitudes h_a, h_b and h_c, and let P be a point inside (or on the boundary of) the triangle. Show that

$$PA + PB + PC \geqslant \frac{2}{3}(h_a + h_b + h_c)$$

with equality if and only if ΔABC is equilateral and P is its center.

1308. *Proposed by Nicholas A. Martin, Indiana University, Bloomington.*

Let $\mathbf{N}$ be the set of natural numbers $\{1,2,3,\dots\}$, let $g\colon \mathbf{N} \to \mathbf{N}$ be a bijection, and $a \in \mathbf{N}$ be an odd number.

a. Prove that there is no function f such that $f(f(n)) = g(n) + a$ for all $n \in \mathbf{N}$.

b^*. What if a is even?

1309. *Proposed by Edward Kitchen, Santa Monica, California.*

Let $0.a_1a_2a_3a_4a_5a_6\dots$, be a period six decimal expansion which determines six distinct lattice points (a_i, a_{i+1}), $i = 1,2,3,4,5,6$, with subscripts taken modulo 6. Prove that these points lie pairwise symmetrically on a central conic (ellipse or hyperbola) if and only if $a_i + a_{i+3} = b$, $i = 1,2,3$, for some positive integer b. (See Problem 1248 and its solution, *Mathematics Magazine* (October 1987) p. 245.)

We invite readers to submit problems believed to be new and appealing to students and teachers of advanced undergraduate mathematics. Proposals should be accompanied by solutions, if at all possible, and by any other information that will assist the editors and referees. A problem submitted as a Quickie should have an unexpected, succinct solution. An asterisk (*) next to a problem number indicates that neither the proposer nor the editors supplied a solution.

Solutions should be written in a style appropriate for Mathematics Magazine. Each solution should begin on a separate sheet containing the solver's name and full address.

Solutions and new proposals should be mailed in duplicate to Loren C. Larson, Department of Mathematics, St. Olaf College, Northfield, MN 55057.

1310. *Proposed by C. B. Khare, Cambridge, England.*

Show that $\sum_{n=1}^{\infty} 1/(\operatorname{lcm}\{1,2,3,\dots,n\})^{\delta}$ converges for all $\delta > 0$. ($\operatorname{lcm}\{1,2,3,\dots,n\}$ is the least common multiple of the set $\{1,2,3,\dots,n\}$.)

1311. *Proposed by Mihály Bencze, Braşon, Romania.*

Let $0 < m \leqslant x_1, x_2, \dots, x_{2n+1} \leqslant M$. Prove that

$$(M-m)^2 + 4Mm\left(\sum_{k=1}^{2n+1} x_k\right)\left(\sum_{k=1}^{2n+1} \frac{1}{x_k}\right) \leqslant (2n+1)^2(M+m)^2.$$

Quickies

Answers to the Quickies are on page 325.

Q739. *Proposed by Murray S. Klamkin, University of Alberta, Canada.*

Determine the largest value of the constant k such that the inequality

$$(x_1 + x_2 + \cdots + x_n)^2(x_1x_2 + x_2x_3 + \cdots + x_nx_1) \geqslant k\left(x_1^2x_2^2 + x_2^2x_3^2 + \cdots + x_n^2x_1^2\right)$$

is valid for all $x_1, x_2, \dots, x_n \geqslant 0$.

Q740. *Proposed by W. O. Egerland and C. E. Hansen, University of Baltimore, Maryland.*

Show that the "Little" Fermat Theorem, $a^p - a \equiv 0 \pmod p$, is a consequence of the Fundamental Theorem of Algebra.

Q741. *Proposed by C. B. Khare, Cambridge, England.*

Suppose $(a_i)_{i=1}^{\infty}$ is a strictly increasing sequence of natural numbers. Does $\sum_{i=1}^{\infty} 1/\operatorname{lcm}\{a_i, a_{i+1}\}$ necessarily converge? ($\operatorname{lcm}\{a, b\}$ denotes the least common multiple of a and b.)

Solutions

An infinite triangular array

1277. *Proposed by Underwood Dudley, DePauw University.* **December 1987**

Determine the next row in the following triangular array of numbers.

```
                              2
                            2   2
                          2   1   2
                        2   0   0   2
                      2   6   5   6   2
                    2   6   4   4   6   2
                  2   6   3   2   3   6   2
                2   6   2   0   0   2   6   2
              2   6   1   9   8   9   1   6   2
            2   6   0   8   6   6   8   0   6   2
          2   6  12   7   4   3   4   7  12   6   2
        2   6  12   6   2   0   0   2   6  12   6   2
      2   6  12   5   0  12  11  12   0   5  12   6   2
    2   6  12   4  14  10   8   8  10  14   4  12   6   2
  2   6  12   3  13   8   5   4   5   8  13   3  12   6   2
2   6  12   2  12   6   2   0   0   2   6  12   2  12   6   2
```

I. *Solution by J. G. Mauldon, Amherst College, Massachusetts.*

For $r = 1, 2, 3, \ldots$ define $T_r = \langle j(j+1) \pmod{r+2},\ j = 1, 2, \ldots, r\rangle$. The 16 rows of the given array are $T_1, T_2, \ldots, T_{16}$, so presumably the "next row" is T_{17}, namely,

2 6 12 1 11 4 18 15 14 15 18 4 11 1 12 6 2

(but see the Note below).

Here is a method of attack on the problem. Each row parallel to one of the two boundaries of the (potentially infinite) given triangular array consists of a sequence of descending arithmetic progressions which are prevented from reaching negative values by the strategic addition of a correction at suitable places. On the rth horizontal row this correction, applied only when necessary, takes the constant value $r+2$, clearly indicating that the rth horizontal row is reduced $(\text{mod } r+2)$. Calculating the second difference in the rth horizontal row we find $(t_{j+1} - t_j) - (t_j - t_{j-1}) \equiv 2 \pmod{r+2}$ which is constant, showing that t_j is quadratic in j. Extrapolating backward we find $(t_{-1}, t_0, t_1) = (0, 0, 2)$, showing that $t_j = j(j+1)$.

Note. Since the array has been produced by a potentially deceitful composer, we are not strictly speaking justified in assuming (by Occam's razor) that the simplest solution is the correct one. Replacement, in the first line of the solution, of $j(j+1)$ by $j(j+1) + 23(17!)\binom{r+2}{j-9}$ yields a triangular array whose first 16 rows are $T_1, T_2, \ldots, T_{16}$, except that the central element of the "next row" T_{17} is 18 instead of 14.

II. *Solution by Kevin Brown, Kent, Washington.*

Denote the location of each array element by its diagonal coordinates (m, n), where m signifies the m-th diagonal descending left to right and n signifies the n-th diagonal descending right to left. Every value in the array is in the range from 0 to the sum of its coordinates, $m+n$. This leads us to suppose that the array values are given by some formula modulo $m+n+1$. We see that the formula must be symmetric in m and n. With a little experimentation we find that the given array is reproduced exactly by the formula

$$a_{m,n} = m^2 + mn + n^2 - 1 \pmod{n+n+1}.$$

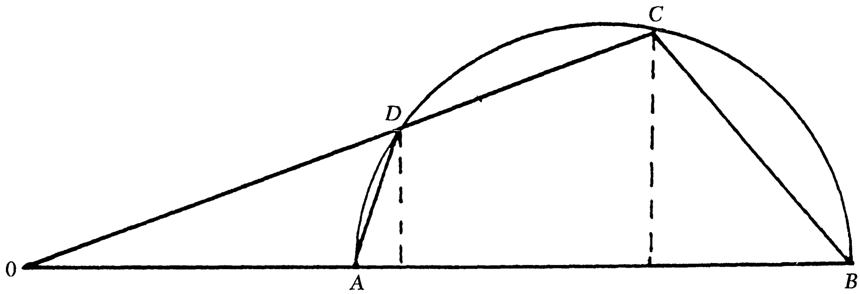
Also solved by Fred Arand, G. Behforooz, Ada Booth, Hollie L. Buchanan II and William Y. Kerr (students), H. Demir, David Earnshaw, Bob Ehren and Greg Gearey (students), Stephen C. Fiete and Paul G. Sutter (students), Tony Fitzgerald (England), The George Washington University Problem Group, Kathleen Goto (student), Constance J. Goutziers, Jerrold W. Grossman, Aparna W. Higgins, Paul G. Howard (student), Kevin Iga (student), Farhood Pouryoussefi Kermany (Iran), L. Kuipers (Switzerland), J. P. Lambert, David M. Larue, Rebecca Lee, Felix Leitner, Carl Libis, Katherine Muhs, G. Nonay, David Pollack and Arjun Kapur (students), Rolf Rosenkranz (West Germany), Joseph Sardinha, Jr., Marlow Sholander, N. J. Taber, Gail Watson, Samuel Yates, and the proposer.

One solution consisted of a row of zeros, and several solutions were just as above except that '18' was replaced by 14, 16, 9, 1, or -1.

Quadrilateral inscribed in semicircle **December 1987**

1278. *Proposed by Howard Eves, University of Central Florida, Orlando.*

If O is a given point on the prolongation of diameter BA of a given semicircle, and if ODC is a secant cutting the semicircle in D and C, prove that quadrilateral $ABCD$ has maximum area when the orthogonal projection of DC on AB is equal to the radius of the semicircle.



Solution by Joseph Sardinha, Jr., Bristol, Rhode Island.

Let Q be the center of the circle. We may assume that the radius of the circle is unity. Let $2e = CD$, $k = OQ$, and $\theta = \angle COB$.

From the diagram, $e = \sqrt{1 - k^2\sin^2\theta}$, $OD = k\cos\theta - e$, $OC = k\cos\theta + e$, $OA = k - 1$, and $OB = k + 1$. Thus,

$$\begin{aligned}\text{Area}(ADCB) &= \tfrac{1}{2}OB \cdot OC\sin\theta - \tfrac{1}{2}OA \cdot OD\sin\theta \\ &= \tfrac{1}{2}\sin\theta((k+1)(k\cos\theta + e) - (k-1)(k\cos\theta - e)) \\ &= k\sin\theta(e + \cos\theta) \\ &= k\sin\theta\sqrt{1 - k^2\sin^2\theta} + k\sin\theta\cos\theta.\end{aligned}$$

Taking the derivative and setting it equal to zero yields

$$(2e\cos\theta - 1)(e + \cos\theta) = 0.$$

From this we see that the maximum occurs when $2e\cos\theta = 1$.

Also solved by The University of Arizona Problem Solving Group, Wadie A. Bassali (Kuwait), J. Binz (Switzerland), Chico Problem Group, H. Demir (Turkey), Jordi Dou (Spain), Ragnar Dybvik (Norway), Ken Gaal, Michael Golomb, Lee O. Hagglund, Francis M. Henderson, Richard A. Jacobson, Vaclav Konecny, L. Kuipers (Switzerland), Neela Lakshmanan, N. J. Lord (England), Kee-Wai Lau (Hong Kong), William A. Newcomb, J. Pfaendtner, Seshadri Sivakumar (Canada), J. M. Stark, Harry Weingarten, and the proposer.

All of the solutions were based on analytical techniques. Is there a purely geometric solution? *Lakshmanan* showed that the result is true, as well, of a semiellipse whose end points are on one of its principal diameters, and obtained a corresponding result when O is on *any* diameter of the ellipse.

Odd binomial coefficients **December 1987**

1279. *Proposed by Abraham P. Hillman, University of New Mexico.*

For nonnegative integers n, let $G(n)$ be the number of integers k such that $\binom{n-k}{k}$ is odd (and $0 \leqslant 2k \leqslant n$).

a. Express $G(2n)$ and $G(2n+1)$ in terms of $G(n)$ and $G(n-1)$.

b. Find $G(2^{1887})$.

(A necessary and sufficient condition using base 2 numerals of m and k for $\binom{m}{k}$ to be odd is known and can be conjectured and proved with the help of

$$(x+1)^{2^a} \equiv x^{2^a} + 1 (\text{mod}\, 2).)$$

Solution by Lorraine L. Foster, California State University, Northridge.

We note that $G(0) = 1 = G(1)$. We shall prove that:

(i) $G(2n+1) = G(n)$ and $G(2n) = G(n) + G(n-1)$, for $n \geqslant 1$, and

(ii) $G(2^v) = v + 1$ (and, in particular, $G(2^{1887}) = 1888$).

Let $m = a_t a_{t-1} \cdots a_0$, $r = b_t b_{t-1} \cdots b_0$ be binary representations of m and r. It is known (cf. D. E. Knuth, *The Art of Computer Programming*, v.1, *Fundamental Algorithms*, Addison-Wesley, 1973, p. 68) that

$$\binom{m}{r} \equiv \prod_{i=0}^{t} \binom{a_i}{b_i} \quad (\text{mod}\, 2) \tag{1}$$

(where $\binom{0}{1} = 0$). Let $\binom{m}{r}_2$ denote the least residue of $\binom{m}{r}$ (mod 2). Then

$$G(n) = \sum_{2k \leqslant n} \binom{n-k}{k}_2.$$

Using (1), it is easy to verify that

$$\binom{2u}{2k+1} \equiv 0, \qquad \binom{2u}{2k} \equiv \binom{u}{k},$$

and

$$\binom{2u+1}{2k} \equiv \binom{u}{k} \equiv \binom{2u+1}{2k+1} \quad (\text{mod}\, 2).$$

It follows that

$$G(2n+1) = \sum_{k \leqslant n} \binom{2n+1-2k-1}{2k+1}_2 + \sum_{k \leqslant n} \binom{2n+1-2k}{2k}_2$$

$$= 0 + \sum_{k \leqslant n} \binom{n-k}{k}_2 = G(n),$$

$$G(2n) = \sum_{k<n} \binom{2n-2k-1}{2k+1}_2 + \sum_{k\leqslant n} \binom{2n-2k}{2k}_2$$
$$= \sum_{k<n} \binom{n-k-1}{k}_2 + \sum_{k\leqslant n} \binom{n-k}{k}_2 = G(n-1) + G(n),$$

for $n \geqslant 1$. This establishes (i).

Since $G(1) = 0 + 1 = G(2^0)$ it follows inductively from (i) that $G(2^v - 1) = G(2^{v-1} - 1) = 1$ and $G(2^v) = G(2^{v-1}) + G(2^{v-1} - 1) = v + 1$, as asserted in (ii).

Also solved by Seung-Jin Bang (Korea), Ada Booth, Fred Dodd, Y. H. Harris Kwong, Ray McClanahan, Tim McMillan (student), Shirley Wakin, and the proposer.

Floor function identity **December 1987**

1280. *Proposed by Donald E. Knuth, Stanford University.*

Prove that

$$\left\lfloor \frac{m^2}{n} \right\rfloor + \sum_{k=0}^{m-1} \left(\left\lfloor \frac{k}{n} \right\rfloor - \left\lfloor \frac{m+k}{n} \right\rfloor \right) = \left\lfloor \frac{\min(m \bmod n, (-m) \bmod n)^2}{n} \right\rfloor$$

for all positive integers m and n. (Here $\lfloor x \rfloor$ denotes the greatest integer $\leqslant x$ and $m \bmod n = m - \lfloor m/n \rfloor n$.)

Solution by H.-J. Seiffert, Berlin, West Germany.

Choose the integers j and r, such that $m = jn + r$, $j \geqslant 0$ and $0 \leqslant r < n$. Then

$$\left\lfloor \frac{m^2}{n} \right\rfloor = jm + jr + \left\lfloor \frac{r^2}{n} \right\rfloor,$$

$m (\bmod n) = r$ and $(-m) (\bmod n) = n - r$, and

$$\sum_{k=0}^{m-1} \left(\left\lfloor \frac{m+k}{n} \right\rfloor - \left\lfloor \frac{k}{n} \right\rfloor \right) = jm + \sum_{i=m}^{m+r-1} \left\lfloor \frac{i}{n} \right\rfloor.$$

Now the stated equation becomes

$$\sum_{i=m}^{m+r-1} \left\lfloor \frac{i}{n} \right\rfloor = jr + \left\lfloor \frac{r^2}{n} \right\rfloor - \left\lfloor \frac{\min(r, n-r)^2}{n} \right\rfloor. \tag{1}$$

If $2r \leqslant n$, then (1) follows immediately from

$$j = \left\lfloor \frac{m}{n} \right\rfloor \leqslant \left\lfloor \frac{i}{n} \right\rfloor \leqslant j + \left\lfloor \frac{2r-1}{n} \right\rfloor = j,$$

valid for $i \in \{m, m+1, \ldots, m+r-1\}$.

Now assume that $n < 2r$. Then an easy computation yields

$$\sum_{i=m}^{m+r-1} \left\lfloor \frac{i}{n} \right\rfloor = \sum_{i=m}^{m+n-r-1} \left\lfloor \frac{i}{n} \right\rfloor + \sum_{i=m+n-r}^{m+r-1} \left\lfloor \frac{i}{n} \right\rfloor$$
$$= j(n-r) + (j+1)(2r-n) = jr + 2r - n.$$

This completes the proof of (1), since

$$\left\lfloor \frac{(n-r)^2}{n} \right\rfloor = n - 2r + \left\lfloor \frac{r^2}{n} \right\rfloor.$$

Also solved by Rich Bauer, J. Binz (Switzerland), Ada Booth, Jesse I. Deutsch, Y. H. Harris Kwong, Garrett R. Vargas (student), Paul J. Zwier, and the proposer.

Acute dihedral angles **December 1987**

1281. *Proposed by Murray S. Klamkin, University of Alberta.*

a. Determine the least number of acute dihedral angles in a tetrahedron.

*b. Generalize the result for an n-dimensional simplex. Here a dihedral angle is the supplement of the angle between outward normals to two $(n-1)$-dimensional faces of the simplex.

a. *Solution by the proposer.*

There is at least one vertex of the tetrahedron such that its corresponding face angles are all acute. Otherwise, since the sum of two face angles at a vertex is greater than the third face angle, the sum of all face angles would be greater than 4π. However, since there are four faces, this sum must be equal to 4π.

We now give two lemmas obtained from the following two laws of cosines from spherical trigonometry for the face angles a, b, c, and the opposite dihedral angles, respectively, of a trihedral angle:

$$\sin b \sin c \sin A = \cos a - \cos b \cos c, \text{ etc.,} \tag{1}$$

$$\sin B \sin C \cos a = \cos A + \cos B \cos C, \text{ etc.,} \tag{2}$$

LEMMA 1. *If $\pi/2 > a \geqslant b \geqslant c$, then from* (1), *$B$ and C are acute.*

LEMMA 2. *If $A \geqslant B \geqslant \pi/2 > C$, then by* (2), *$a, b \geqslant \pi/2 > c$.*

In a tetrahedron $PQRS$, we can take P as a vertex with all acute face angles. Then by Lemma 1, we can take PQ and PS as the edges of two acute dihedral angles. We now assume that there are no more acute dihedral angles and obtain a contradiction. By Lemma 2 applied to the trihedral angles at vertices Q and S, it follows that angles PQS and PSQ are both non-acute. Since this is impossible, there are always at least three acute dihedral angles in any tetrahedron.

b. *Partial solution by L. P. Pook, Glasgow, Scotland.*

For finite n the $n(n-1)/2$ dihedral angles of a regular n-dimensional simplex are acute, so this is an initial upper bound.

A lower value for the upper bound is obtained by constructing a low altitude right hyper-pyramid, vertex V, with a regular $(n-1)$-dimensional simplex as base. The altitude is chosen such that the dihedral angles between pyramidal faces are obtuse; the remaining n dihedral angles between the base and the pyramidal faces must be acute.

Distorting the pyramid by moving V, in a hyperplane parallel to the base, to a position such that at least one dihedral angle at the base becomes obtuse results in at least an equal number of dihedral angles between pyramidal faces becoming acute. In view of this it is conjectured that n is indeed the required minimum number of acute dihedral angles.

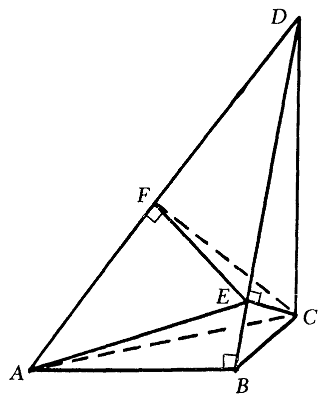
A trinity of orthoschemes **December 1987**

1282. *Proposed by H. S. M. Coxeter, University of Toronto.*

In Euclidean space, a tetrahedron is called an orthoscheme if its faces consist of four right-angled triangles. How can such a tetrahedron be dissected (by means of two cutting planes) into three pieces each of which is an orthoscheme? Can the three pieces all have the same volume? Are they then congruent?

Solution by the proposer.

By trying the various possibilities, one soon finds that the four faces must be arranged so that two right angles occur at each of two vertices, say B and C, yielding a tetrahedron $ABCD$ such that the three edges AB, BC, CD are mutually perpendicular (that is, parallel to the three Cartesian axes). In other words, the two skew lines AB and CD are such that each lies in a plane perpendicular to the other.



The two desired "cutting planes" are ACE and CEF, where CE is the perpendicular from C to BD and EF is the perpendicular from E to AD. Since the two skew lines CE and AD are perpendicular in the above sense, all the four angles at F (in the planes ABD and ACD) are right angles. Therefore the three tetrahedra $ABEC$, $AFEC$, $DFEC$ are orthoschemes.

Since the volume of an orthoscheme is one-sixth of the product of its three mutually perpendicular edges, the three smaller orthoschemes will all have the same volume if

$$AB \times BE \times EC = AF \times FE \times EC = DF \times FE \times EC,$$

that is, if F is the midpoint of AD and

$$AB \times BE = AF \times FE.$$

The right angles ABC, ABD, ACD, BCD, AEC, BEC, AFE, AFC, FEC, DFE DFC, DEC provide (by means of Pythagoras) a somewhat redundant set of equations for the various edge-lengths. We conclude that, if $BE = 1$, we have also

$$EF = 1, \quad CD = \sqrt{2}, \quad AB = BC = CF = AF = FD = \sqrt{3},$$
$$AE = DE = 2, \text{ and } AC = CD = \sqrt{6}.$$

Thus the three smaller orthoschemes are all congruent: $ABEC$ and $DFEC$ are *directly* congruent because they are images of $AFEC$ by reflections in the two cutting planes.

Also solved by Richard A. Jacobson.

A dispersed set of points in a hypercube **December 1987**

1283. *Problem due to Leo Moser (1921–1970), submitted by William O. J. Moser, McGill University.*

Show that the maximal number of points which can be located on the 4-dimensional unit cube such that all mutual distances are $\geqslant 1$ is 17. Furthermore, the only configuration yielding this number is the center and the 16 vertices.

Solution by Michael Golomb, Purdue University, West Lafayette.

To avoid fractions we assume the cube Q is of length 2. We must show that the maximal number of points which can be located on Q such that all mutual distances are $\geqslant 2$ (we call such a set *dispersed*) is 17. Introduce cartesian coordinates such that the center C of Q is at the origin and Q is the set of points (x_1, x_2, x_3, x_4) for which $-1 \leqslant x_i \leqslant 1$ $(i = 1, 2, 3, 4)$. Divide Q into 16 cubes $q_1, q_2, \dots, q_{16}$, where each q_k is given by inequalities $\epsilon_i \leqslant x_i \leqslant \epsilon_i + 1$ $(i = 1, \dots, 4)$ with ϵ_i equal to -1 or 0. If (y_1, y_2, y_3, y_4) is a vertex of some q_k then (z_1, z_2, z_3, z_4) is the diametrically opposite vertex if $|z_i - y_i| = 1$ $(i = 1, \dots, 4)$. There are no two points in any q_k whose distance is > 2, and if the distance is 2, the points must be a pair of diametrically opposite vertices.

By omitting some of the boundary points of each q_k we can divide $Q \setminus C$ into nonclosed cubes $q_1^-, \dots, q_{16}^-$ such that $Q \setminus C$ is the disjoint union of these q_k^- and such that no q_k^- has diametrically opposite vertices. If S is a dispersed set on Q then no q_k^- can contain more than one point of S, thus S has no more than $16 + 1$ points. On the other hand the set S_* consisting of the center and the vertices of Q contains 17 points and is dispersed. If (x_1, x_2, x_3, x_4) and (y_1, y_2, y_3, y_4) are distinct vertices of Q (each x_i and y_i is $+1$ or -1), their distance is $(\Sigma(x_i - y_i)^2)^{1/2} \geqslant \max|x_i - y_i| = 2$ and their distance from C is $(\Sigma x_i^2)^{1/2} = 2$.

J. G. Mauldon proved more generally that any set of 16 or more points on the 4-dimensional unit cube with mutual distances $\geqslant 1$ must be a subset of the center and the 16 vertices.

Inequality when f and $1/f$ are both convex **December 1987**

1284. *Proposed by H.-J. Seiffert, Berlin, West Germany.*

Let $f \in C^2[a, b]$ be strictly positive. Assuming that f and $1/f$ are both convex, prove that

$$\int_a^b \left(\frac{f'(x)}{f(x)} \right)^2 dx \leqslant \frac{1}{b-a} \frac{(f(a) - f(b))^2}{f(a)f(b)}.$$

Solution by Kee-Wai Lau, Hong Kong.

For $a \leqslant x \leqslant b$ let

$$g(x) = f(x) - f(a) - \frac{(f(b) - f(a))(x - a)}{b - a}.$$

We have $g(a) = g(b) = 0$ and $g''(x) = f''(x) \geqslant 0$. Thus, $g(x)$ is convex and $g(x) \leqslant 0$.

Now

$$\int_a^b \left(\frac{f'(x)}{f(x)} \right)^2 dx - \frac{1}{b-a} \frac{(f(a) - f(b))^2}{f(a)f(b)}$$
$$= \int_a^b \frac{f'(x)}{f^2(x)} \left(f'(x) - \frac{f(b) - f(a)}{b - a} \right) dx$$

$$= \int_a^b \frac{f'(x)}{f^2(x)} d(g(x))$$

$$= \left[\frac{f'(x)}{f^2(x)} g(x) \right]_a^b + \int_a^b g(x)(1/f(x))'' \, dx$$

$$= \int_a^b g(x)(1/f(x))'' \, dx.$$

Since $1/f(x)$ is convex, and $g(x) \leqslant 0$, the integrand of the last integral is nonpositive and hence the integral itself is nonpositive. This leads immediately to the result of the problem. Note also that equality holds if and only if f is linear.

Also solved by Jeff Benedict, Michael Golomb, Eugene Levine, Stephen Noltie, and the proposer.

Levine showed that the result is a consequence of the following inequality due to Chebychev (cf. Hardy, Littlewood, Pólya, *Inequalities*, Cambridge University Press, 1952): Let $G(x)$ and $H(x)$ be continuous functions on $[a, b]$ where $G(x)$ is nonincreasing and $H(x)$ is nondecreasing. Then

$$(b-a)\int_a^b G(x)H(x)\,dx \leqslant \int_a^b G(x)\,dx \int_a^b H(x)\,dx.$$

Point-covering problem **December 1987**

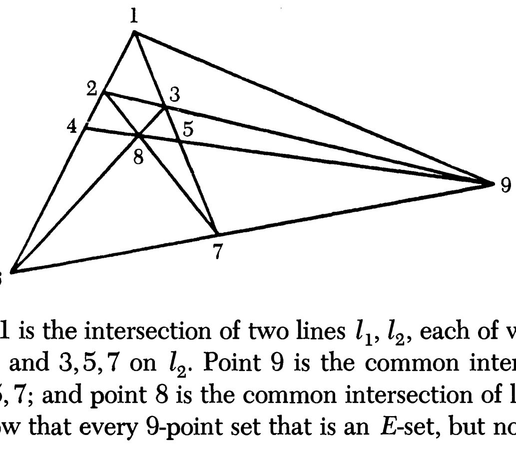
1285. *Proposed by Paul Erdös and László Lovász, Hungarian School of Sciences.*

Let n be sufficiently large, and let there be given n points in the plane so that every five of them can be covered by two lines. Prove that then all the points can be covered by two lines. Show that this is false for $n = 8$, but holds for $n \geqslant 9$.

Solution by Michael Golomb, Purdue University, West Lafayette.

To facilitate the presentation we say that a point set S is an L-set if all of its points are covered by one or two lines; and S is an E-set if every subset of 5 points is covered by one or two lines. Clearly S is an E-set if and only if in every subset of 5 points 3 points are collinear.

The problem asserts that every E-set with at least 9 points is an L-set. This is false, as shown in the following figure. There we have an E-set with 9 points which clearly is not an L-set.



In this set, point 1 is the intersection of two lines l_1, l_2, each of which covers 3 more points: 2, 4, 6 on l_1 and 3, 5, 7 on l_2. Point 9 is the common intersection of the lines joining 2, 4, 6 to 3, 5, 7; and point 8 is the common intersection of lines 2, 4, 6 to 7, 5, 3. It is not hard to show that every 9-point set that is an E-set, but not an L-set, is of this structure.

We now show that every E-set with more than 9 points is an L-set. The proof is based on the following Lemma.

LEMMA. *Every 9-point subset S_9 of an E-set S with more than 9 points is an L-set.*

Proof. If S_9 is not an L-set then it is of the structure described above. Let 10 be a point in $S \setminus S_9$. If 10 lies on none or only one of the lines determined by the pairs of points of S_9, then clearly S cannot be an E-set. So we may consider only cases where point 10 is the intersection of two such lines. Of all these possibilities only two somewhat different cases need be considered. 10 is the intersection of the lines determined by $\{2,5\}$ and $\{6,9\}$ or by $\{1,8\}$ and $\{6,9\}$. In the first case $\{1,2,8,9,10\}$, in the second case $\{1,2,5,9,10\}$ is a subset of S which is not an L-set, contradicting the hypothesis.

Assume now S_{10} is an E-set with 10 points that is not an L-set. We will show that this is impossible by finding for every such set a 5-point subset that is not an L-set. We use $\mathscr{L}$ for the family of L-sets. By the Lemma S_{10} has a subset $S_9 \in \mathscr{L}$; let C denote the point in $S_{10} \setminus S_9$. Disregarding the trivial cases where either 9 or 8 of the points of S_9 are collinear, there are two uniquely determined lines $\underline{a}$ and $\underline{b}$ such that $S_9 \subset \underline{a} \cup \underline{b}$. Let A_i denote points of S_9 on $\underline{a}$, B_i points of S_9 on $\underline{b}$. If the point $P \in S_9$ is both on $\underline{a}$ and $\underline{b}$ it will be considered as one of the A_i alone. To express that P, Q, R are collinear points we write $\overline{PQR}$. $\overline{A_iA_jB_k}, \overline{A_iA_jC}, \overline{B_iB_jA_k}, \overline{B_iB_jC}$ are not possible for $i \neq j$ by our assumptions. $\overline{A_iB_jC}$ is possible, and if this is the case we say the pair (A_i, B_j) is improper, otherwise proper. We will show that for any given triple $(\underline{a}, \underline{b}, C)$ there are points A_1, A_2, B_1, B_2 such that the 4 pairs (A_i, B_j) $(i, j = 1,2)$ are proper, which then implies $(A_1, A_2, B_1, B_2, C) \notin \mathscr{L}$.

To prove the assertion it suffices to consider the cases where all the A_i or all the B_j are components of improper pairs. The largest number of improper pairs is 4, say $(A_1, B_3), (A_2, B_4), (A_3, B_1), (A_4, B_2)$ are such; then $(A_1, A_2, B_1, B_2, C) \notin \mathscr{L}$. If A_1, A_2, A_3 are the points of S_9 on $\underline{a}$ and $(A_1, B_3), (A_2, B_4), (A_3, B_5)$ are improper, then $(A_1, A_2, B_1, B_2, C) \in \mathscr{L}$. If A_1, A_2 are the points of S_9 on $\underline{a}$ and $(A_1, B_3), (A_2, B_4)$ are improper then $(A_1, A_2, B_1, B_2, C) \notin \mathscr{L}$.

Having proved that every E-set with 10 points is an L-set, let S be now an E-set with more than 10 points. We may assume that S contains a subset S_{10} of 10 points which is not covered by a line or by a line and a point. S_{10} is an L-set, covered by two uniquely determined lines l_1, l_2. It follows immediately that every point of $S \setminus S_{10}$ must be on l_1 or l_2, hence all of S is an L-set and the proof is complete.

Also solved by Irl C. Bivens and Benjamin G. Klein, and David Callan. There was one incomplete solution.

Upper bound for variation **December 1987**

1286. *Proposed by Paul C. Rosenbloom, Teachers College, Columbia University.*

Suppose that $F(x)$ is continuous and nondecreasing and g is continuous and nonincreasing and positive for $x \geqslant 0$ so that the set E of x such that $F(x+1) - F(x) < g(x)$ is a union, $\bigcup_{n=0}^{\infty}(a_n, b_n)$, of disjoint open intervals, $a_n < b_n \leqslant a_{n+1}$ for all $n \geqslant 0$. Show that $V_F(E)$, the variation of F on E,

$$V_F(E) = \sum_{n=0}^{\infty} (F(b_n) - F(a_n)),$$

satisfies $V_F(E) \leqslant \sum_{n=0}^{\infty} g(n)$.

Solution by the proposer.

Let $E_n = E \cap [n, n+1)$. If E_n is not empty, let $x = a_k$ be the greatest lower bound of E_n. Then we have

$$V_F(E_n) \leqslant F(n+1) - F(a_k) \leqslant F(a_k+1) - F(a_k) \leqslant g(a_k) \leqslant g(n),$$

so that

$$V_F(E) = \sum_{n=0}^{\infty} V_F(E_n) \leqslant \sum_{n=0}^{\infty} g(n).$$

Answers

Solutions to the Quickies on p. 315.

A739. Without loss of generality we can let $x_1 + x_2 + \cdots + x_n = 1$. Then letting $n = 2$, $x_1 = x_2 = 1/2$, we must satisfy $1/4 \geqslant k/4^2$. Thus, k must be $\leqslant 4$. For $k = 4$, the inequality becomes

$$x_1x_2(1 - 4x_1x_2) + x_2x_3(1 - 4x_2x_3) + \cdots + x_nx_1(1 - 4x_nx_1) \geqslant 0$$

and this inequality holds since $\max(x_ix_j) = 1/4$. There is equality if and only if one variable equals 1 or else two successive variables equal 1/2, or trivially, if all the variables are zero.

A740. Let f be a polynomial of degree a, p a prime, and f^p the pth iterate of f. The equation $f^p(z) - z = 0$ has a^p solutions, a of which are the fixed points of f. Each of the remaining $N = a^p - a$ solutions is a p-periodic point of f, so that N counts the number, say m, of p-periodic orbits of f. Hence, $N = mp = a^p - a$ and the congruence follows.

A741. We have $(a_{i+1} - a_i)\ \mathrm{lcm}\{a_i, a_{i+1}\} \geqslant \gcd\{a_i, a_{i+1}\}\ \mathrm{lcm}\{a_i, a_{i+1}\} = a_ia_{i+1}$, and, therefore,

$$\frac{1}{\mathrm{lcm}\{a_i, a_{i+1}\}} \leqslant \frac{a_{i+1} - a_i}{a_ia_{i+1}} = \frac{1}{a_i} - \frac{1}{a_{i+1}}.$$

Thus the series converges, by the comparison test, because

$$\sum_{i=1}^{\infty}\left(\frac{1}{a_i} - \frac{1}{a_{i+1}}\right)$$

is a convergent telescoping series.

REVIEWS

PAUL J. CAMPBELL, *editor*
Beloit College

Assistant Editor: Eric S. Rosenthal, West Orange, NJ. Articles and books are selected for this section to call attention to interesting mathematical exposition that occurs outside the mainstream of the mathematics literature. Readers are invited to suggest items for review to the editors.

Ascher, Marcia, Graphs in cultures: A study in ethnomathematics, *Historia Mathematica* 15 (1988) 201-227.

Like many other cultures, the Malekula of Vanuatu (formerly the New Hebrides) have a folk tradition of continuous figure-tracing. In fact, passage to their Land of the Dead requires the deceased to complete a partially-traced figure. Ascher classifies the Malekula patterns, analyzes them in terms of a process algebra, and investigates symmetries.

Peterson, Ivars, Tiling to Infinity, *Science News* 134 (16 July 1988) 42.

Two IBM researchers, including a ceramics expert, together with a crystal theorist at the University of Pennsylvania, have worked out sufficient conditions for a tiling of the plane with Penrose's diamonds. As long as the tiling is built up with every vertex figure being one of the eight they enumerate, no "defect" can arise. Details can be found in *Physical Review Letters* (20 June 1988).

Weiss, Eric A. (ed.), *A Computer Science Reader: Selections from Abacus*, Springer-Verlag, 1988; xvi + 447 pp.

This book, which contains selected articles from "the first 3 1/2 years" of *Abacus*, arrived the same week as the final issue of *Abacus*. *Abacus*, edited by Tony Ralston (SUNY Buffalo), was a magazine on computing and computer science, modeled on *Scientific American* and directed to the general informed public. In case you didn't subscribe—and especially in case your library didn't—here's a rare opportunity to benefit from the best of *Abacus*. The many selections include history of the electronic digital computer, the practice of software development, the impact of computing on mathematics, data compression, automated reasoning, Fifth Generation computing, and much more. The selections have been retypeset to fit the smaller page size, so some of the elegance of the original layout (with its occasional second color) is lost; and a few long articles have been shortened. (Fellow lovers of *Abacus*: Pray for its resurrection.)

Shasha, Dennis, *The Puzzling Adventures of Dr. Ecco*, Freeman, 1988; xiii + 181 pp, \$16.95, \$9.95 (P).

Here are mathematical puzzles, requiring clever thinking rather than specific mathematical background, clothed in interesting narrative. Solutions are included, except for the 10 cryptograms (solve them and win a T-shirt). A great gift for a student in grades 6-10.

Ribenboim, Paulo, *The Book of Prime Number Records*, Springer-Verlag, 1988; xxiii + 476 pp, $49.80.

Less a compendium of records than a definitive monograph on primes, this book deserves a place on every mathematician's bookshelf. It offers 9 1/2 proofs of the infinity of primes, discusses primality testing, investigates functions defining primes, dwells on the distribution of primes (including twin primes, pseudoprimes, and Carmichael numbers), and also treats special kinds of primes and probabilistic results about primes. Results quoted are keyed to a 100-page bibliography. The only quibble: The book would be closer to the author's ideal of "small and lightweight ... not a bulky brick" if he hadn't prepared the camera-ready copy with half a space between each line.

Rosenfeld, B.A., *A History of Non-Euclidean Geometry: Evolution of the Concept of a Geometric Space*, Springer-Verlag, 1988; xi + 471 pp, $89.

Thorough history of non-euclidean geometry (including Lie groups), translated from the Russian edition (1976), with corrections and supplements.

Levy, David N. L., *Computer Games I*, Springer-Verlag, 1988; xiii + 456 pp, $45.

Reprint of 16 fundamental papers on four games that require intelligence to play well: backgammon, chess, checkers, and Scrabble, plus an extensive bibliography. Shannon's original chess machine article is here, with Samuel's articles on machine learning of checkers, and two articles on optimal doubling in backgammon. The second volume will treat various card games and board games.

Stewart, Ian, The nature of stability, *Speculations in Science and Technology* 10:4 (1987) 310-324.

Few articles have an abstract consisting of two figures (cartoons) and two words ("stable" and "unstable"). The article itself is chock full of figures illustrating dynamical systems and chaos. Says the author: "The name 'chaos' reflects ignorance. ... [T]here is a fundamental mathematical difference between the complex, pseudorandom nature of chaos; and true indeterminacy—whatever that may be."

Hoffman, Paul, *Archimedes' Revenge: The Joys and Perils of Mathematics*, Norton, 1988; viii + 285 pp, $17.95.

Attractive popularization of concepts of mathematics, including quotations from interviews with leading mathematicians. The author is the editor-in-chief of *Discover* and also writes under the pen name "Dr. Crypton."

Peitgen, Heinz-Otto, and Saupe, Dietmar (eds.), *The Science of Fractal Images*, Springer-Verlag, 1988; xiii + 312 pp, $34.

This beautiful book is based on a short course given at SIGGRAPH '87. It includes an introduction to fractals and their applications, a survey of random fractals, an introduction to fantastic fractals, and fractal modeling of real objects.

Hay, Cynthia (ed.), *Mathematics from Manuscript to Print* 1300-1600, Oxford Univ Pr, 1988; viii + 273 pp, $63.

Proceedings of a conference on Renaissance mathematics. A third of the book is devoted to N. Chuquet. Also notable are chapters on Oriental sources of well-known problems in recreational mathematics (D. Singmaster) and on the development of perspective in art and of its mathematical basis (I. Schneider).

Peterson, Ivars, *The Mathematical Tourist: Snapshots of Modern Mathematics*, Freeman, 1988; xv + 240 pp, \$17.95.

"You're entering Mathland ..." Your tourguide admits to not being a native; but as a science journalist for *Science News*, he is a highly-respected interpreter. He interprets mathematics into prose, so almost no equations appear. Much of the material appeared in an earlier incarnation in *Science News*, over the last six years; so the topics are current and likely to interest the layperson: primes, factoring, minimal surfaces (beautiful color plates of these!), knots, higher dimensions, cellular automata, zero-knowledge proofs, and (of course) fractals and chaos.

Williams, Kenneth S., and Hardy, Kenneth, *The Red Book: 100 Practice Problems for Undergraduate Mathematics Competitions*, Integer Press (Box 6613, Station J, Ottawa, Ontario, Canada K2A 3Y7), 1988; ix + 176 pp.

Collection of problems for students training for the Putnam and other mathematical competitions. Hints are given, as well as complete solutions.

Hilton, Peter, and Pedersen, Jean, *Build Your Own Polyhedra*, Addison-Wesley, 1988; 175 pp, \$27.50.

"We have chosen an expository style that should ... make this book accessible—and attractive—to any intelligent person aged between twelve and one hundred." The book begins with folding regular polygons, goes on to constructing flexagons, then gets into the "serious" business of building pop-up polyhedra, dodecahedra, braided platonic solids, and collapsoids.

Gardner, Martin, Crack the problem; win fame, *New York Times* (25 May 1988) Op Ed page.

"The problem" is Fermat's Last Theorem. "Most mathematicians are convinced the theorem is true and eventually will be proved. A minority suspect it is false but believe that the simplest counterexample involves values of a, b and c that have millions of digits. ... I belong to a whimsical third group of people who believe and hope the theorem is undecidable. ... If Fermat's theorem is false, there must be a counterexample, but of course its existence would make the theorem decidable. It follows that if the theorem is Gödel-undecidable, it must be true."

Devlin, Keith, *Mathematics: The New Golden Age*, Penguin, 1988; iv + 287 pp, \$8.95 (P).

"What this book sets out to do is to try to convey to the interested layperson some of the most significant developments that have taken place in mathematics during recent times ... 1960 to 1985." The last year has seen a bumper crop of mathematics books for the general public. The author of this one doesn't mind using equations, infinite series, logarithms, and other notation; so the book is at a relatively high level for "the interested layperson," and it tries to convey serious mathematics. Topics include primality and factoring, undecidability, the class number problem, chaos, simple groups, Hilbert's Tenth Problem, the four-color theorem, Fermat's Last Theorem, the Riemann Hypothesis, knots, and NP-completeness.

Herz-Fischler, Roger, *A Mathematical History of Division in Extreme and Mean Ratio*, Wilfrid Laurier Univ Pr, 1987; xvi + 190 pp, \$Can 65, \$US 75.

Detailed history of the appearance and development of the mathematics of the golden ratio, from Euclid to the 18th century. A second volume will offer the history of the non-mathematical aspects of the subject.

Ekeland, Ivar, *Mathematics and the Unexpected*, Univ of Chicago Pr, 1988; xiii + 146 pp, $19.95.

"The two patron saints of the book are Poincaré and Bergson" (Felix Browder, from the Foreword). Few books have patron saints, but it may be well for one on the mathematics of time to be so blessed. It begins with celestial mechanics, gives Poincaré's contribution, discusses determinism vs. randomness, goes on to stability considerations, then closes with catastrophe theory. An appendix gives details of the Feigenbaum bifurcation.

Aspray, William, and Kitcher, Philip (eds.), *History and Philosophy of Modern Mathematics*, University of Minnesota Pr, 1988; viii + 386 pp, $35, $13.95 (P).

Notable articles: G. H. Moore on the emergence of first-order logic, G. Birkhoff and M. K. Bennett on Klein's Erlanger Programm, J. Dauben on nonstandard analysis, R. Askey on how mathematicians and historians can help each other, and J. V. Grabiner on artificial intelligence.

Hart, Derek, and Croft, Tony, *Modelling with Projectiles*, Wiley, 1988; 152 pp, $39.95.

Bullets, balls of all kinds, javelins, mechanical harvesters, and safe driving on gravel are some of the topics in this interesting collection of mathematical models. Calculus and vector notation are used without apology, and several computer programs are included. Some mathematical expressions could benefit from more open spacing.

Dewdney, A. K., *The Armchair Universe: An Exploration of Computer Worlds*, Freeman, 1988; xiii + 330 pp, $19.95 ($13.95 (P)).

Following in the happy tradition of collecting Martin Gardner's "Mathematical Games" columns into volumes, Freeman here collects Dewdney's "Computer Recreations" columns from *Scientific American*. Addenda to the chapters provide updating from the time of the original publication.

Gerstell, Mimi, *The Trigonometric Travelogue*, Pico Beach Books (Box 67, Revere, MA 02151),1988; x + 192 pp, $14.92.

"I like to look out my eastern window [in Winthrop, Mass.] and know that I am looking towards the iron mines of Mauritania, not towards France." (Hint: Think great circle.) Light text on geodesy—spherical geometry and trigonometry, as applied to the earth. No exercises; a formula summary would be a useful addition.

International Association for the Evaluation of Educational Achievement, *Science Achievement in Seventeen Countries: A Preliminary Report*, Pergamon, 1988; xi + 125 pp, $8 (P).

Assessment of science achievement at ages 10, 14, and in the last year of secondary education. At the first two ages, the U.S. ranked 8th and 14th, respectively, out of 17. For grade 12/13 students taking a science, the U.S. ranked 13th in biology, 11th in chemistry, and 9th in physics, out of 13 countries. For the same subjects in the same order, 98%, 48%, and 89% of U.S. schools had means lower than the lowest school in the highest-ranking country. In all countries, boys score higher than girls (except for biology in Sweden and Hong Kong), with no discernible changes since 1970. The report's conclusion concerning the U.S.: "For a technologically advanced country, it would appear that a reexamination of how science is presented and studied is required."

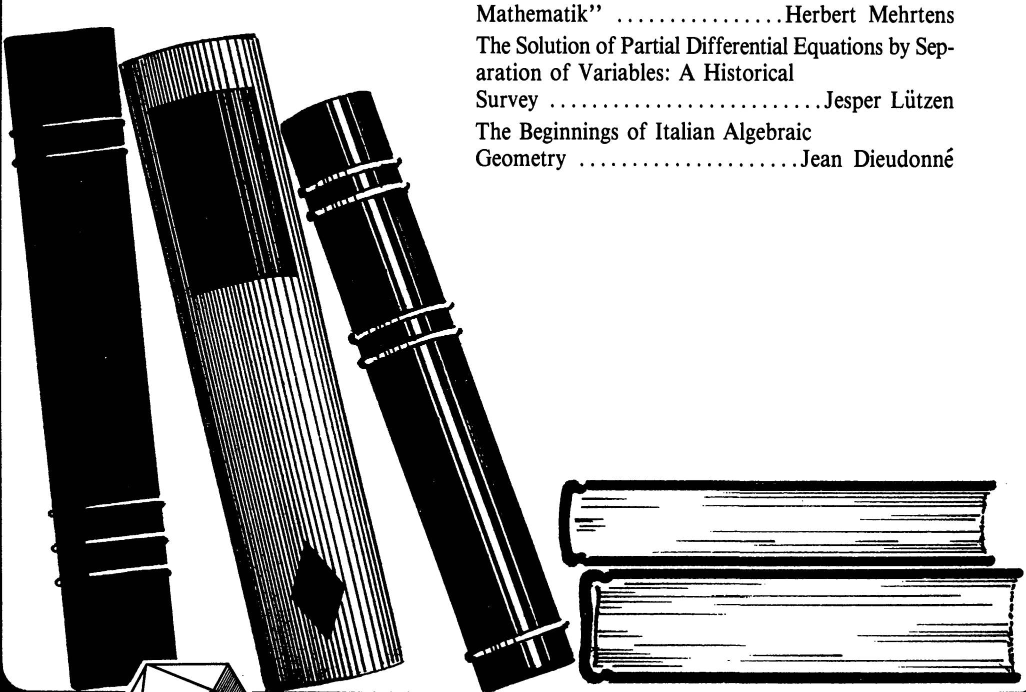
NEWS AND LETTERS

ACKNOWLEDGEMENTS

In addition to our associate editors, the following have assisted the Magazine *by refereeing papers during the past year. We appreciate the time and care they have given.*

Acu, Dumitru, *Sibiu, Romania*
Archibald, Thomas, *Acadia University*
Ash, Robert B., *University of Illinois*
Ayoub, Ayoub B., *Pennsylvania State University, Abington*
Bankoff, Leon, *Los Angeles, CA*
Barbeau, Edward J., *Univ. of Toronto*
Barksdale, Jr., James B., *Western Kentucky University*
Barrett, Lida, *Mississippi State University*
Baxley, John V., *Wake Forest University*
Beals, Richard W., *Yale University*
Beaver, Olga R., *Williams College*
Beezer, Robert A., *Univ. of Puget Sound*
Beineke, Lowell, *Indiana-Purdue University at Ft. Wayne*
Bergstrand, Deborah J., *Williams College*
Bishop, Richard L., *University of Illinois*
Bisztriczky, Ted, *University of Calgary*
Boas, Ralph P., *Northwestern University*
Booth, Ada, *Santa Clara University*
Braden, Bart, *Northern Kentucky University*
Bremser, Priscilla, *Middlebury College*
Brunson, Barry W., *Western Kentucky University*
Buhler, Joe P., *Reed College*
Burckel, Robert B., *Kansas State University*
Cabaniss, Sharon, *San Jose State University*
Cammack, Larry, *Central Missouri State University*
Castellanos, Dario, *Universidad de Carabobo, Valencia, Venezuela*
Chakerian, G.D., *Univ. of California, Davis*
Cicenia, Robert A., *Montvale, NJ*
Conrey, J. Brian, *Oklahoma State Univ.*
Coxeter, H.S.M., *University of Toronto*
Cross, James T., *University of the South*
Cundy, H. Martyn, *Kendal, Cumbria, England*
Davis, Philip J., *Brown University*
deBouvère, Karel L., *Santa Clara Univ.*
Demir, Hüseyin, *Middle East Technical University, Turkey*
DeTemple, Duane W., *Washington State University*
Dixon, Michael, *California St. Univ. at Chico*
Drobot, Vladimir, *Santa Clara Univ.*
Drucker, Thomas, *Dickinson College*
Edgar, Gerald A., *Ohio State University*
Edgar, Hugh M.W., *San Jose State University*
Eisenberg, Bennett, *Lehigh University*
Epp, Susanna, *De Paul University*
Eves, Howard, *Oviedo, FL*
Farris, Frank, *Santa Clara University*
Farris, Steven L., *Ball State University*
Feroe, John, *Vassar College*
Firey, William J., *Oregon State Univ.*
Fisher, Evan, *Lafayette College*
Fisher, J. Chris, *University of Regina*
Foster, Lorraine L., *California State University, Northridge*
Gardner, Martin, *Hendersonville, NC*
Ghandehari, Mostafa, *Santa Clara Univ.*
Gibbs, Richard A., *Fort Lewis College*
Gilbert, William J., *Univ. of Waterloo*
Goldstein, Jerome A., *Tulane University*
Golomb, Solomon, *University of Southern California*
Grassl, Richard M., *University of New Mexico*
Green, Euline "Bo", *Abilene Christian University*
Gretsky, Neil E., *Univ. of California, Riverside*
Groeneveld, Richard A., *Iowa State University*
Grosshans, Frank D., *West Chester Univ.*
Grünbaum, Branko, *Univ. of Washington*
Guy, Richard, *University of Calgary*
Hafner, James, *IBM Research, San Jose, CA*
Hall, James E., *Westminster College*
Halmos, Paul R., *Santa Clara University*
Hamming, R.W., *Naval Postgraduate School*
Hanson, Denis, *University of Regina*
Harary, Frank, *New Mexico State University*
Harrell, Ronald E., *Allegheny College*
Hayashi, Elmer K., *Wake Forest University*
Heinrich, Katherine, *Simon Fraser University*
Henderson, James, *Colorado College*
Henrion, Claudia, *Middlebury College*
Hickling, Fred, *Santa Clara University*
Hillman, A. P., *Albuquerque, NM*
Holden, Herb, *Gonzaga University*
Honsberger, Ross, *University of Waterloo*
Hood, J. Myron, *California Polytechnic State University*
Howard, F.T., *Wake Forest University*
Jackson, Bradley W., *San Jose State University*
Jamison, Robert E., *Clemson University*
Janke, Steven, *Colorado College*
Jepson, Charles H., *Grinnell College*
Johnsonbaugh, Richard F., *DePaul University*
Kalman, Dan, *Rancho Palos Verdes, California*
Katz, Victor J., *Silver Spring, Maryland*

Keller, Edward L., *California State University, Hayward*
Kelly, Leroy M., *Michigan State Univ.*
Klamkin, Murray S., *Univ. of Alberta*
Klosinski, Leonard, *Santa Clara University*
Konhauser, Joseph D.E., *Macalester C.*
Kuzmanovich, James, *Wake Forest Univ.*
Lautzenheiser, Roger G., *Rose-Hulman Institute of Technology*
Levine, Alan, *Franklin & Marshall C.*
Liu, Andy, *University of Alberta*
LoBello, Anthony, *Allegheny College*
Logothetti, Dave, *Santa Clara Univ.*
Loud, Warren S., *Univ. of Minnesota, Minneapolis*
Lutwak, Erwin, *Polytechnic Institute of New York*
Mandelkern, Mark, *New Mexico State University*
Merrill, Kathy, *Colorado College*
Mitchem, John, *San Jose State University*
Monson, Barry, *Univ. of New Brunswick*
Moore, David S., *Purdue University*
Morrill, John E., *DePaul University*
Morrison, John, *Towson State University*
Mugler, Dale, *Santa Clara University*
McCartney, Philip, *Northern Kentucky University*
McCleary, John, *Vassar College*
McDermot, Richard, *Allegheny College*
Nelsen, Roger B., *Lewis & Clark College*
Niven, Ivan, *University of Oregon*
Olinick, Michael, *Middlebury College*
Osofsky, Barbara, *Rutgers University*
Owens, Robert, *Lewis & Clark College*
Pederson, Jean, *Santa Clara University*
Peterson, Bruce, *Middlebury College*
Pfiefer, Richard, *San Jose State Univ.*
Propp, James G., *Berkeley, CA*
Rebman, Kenneth R., *California State University, Hayward*
Redmond, Don, *Southern Illinois Univ.*
Reznick, Bruce, *University of Illinois*
Richmond, Thomas A., *Western Kentucky University*
Rigby, John F., *University College*
Riley, John, *Bloomsburg University*
Robertson, Jack M., *Washington State Univ.*
Roeder, David W., *The Colorado College*
Rosenstein, George M., *Franklin & Marshall College*
Rosier, Ronald, *Georgetown University*
Ross, Peter, *Santa Clara University*
Roth, Richard, *University of Colorado*
Roy, Ranjan, *Beloit College*
Sagan, Hans, *North Carolina State Univ.*
Salwach, Chester J., *Lafayette College*
Samuels, Stephen M., *Purdue University*
Sandefur, James T., *Georgetown Univ.*
Sands, Bill, *University of Calgary*
Sangwine-Yager, Jane, *Davis, CA*
Saunders, Sam C., *Washington State University*
Sawka, John, *Santa Clara University*
Schattschneider, Doris, *Moravian C.*
Schecter, Stephen, *North Carolina State Univ.*
Schoenfeld, Alan H., *Univ. of California, Berkeley*
Schrag, Gerald C., *Central Missouri State University*
Schumer, Peter, *Middlebury College*
Senechal, Marjorie, *Smith College*
Sloughter, Dan, *Furman University*
Small, Lynne B., *University of San Diego*
Smith, Bryan, *University of Puget Sound*
Smith, Martha, *University of Texas at Austin*
Smolarski, Dennis, *Santa Clara Univ.*
Smullyan, Raymond, *Elka Park, NY*
Spiegel, Murray, *East Hartford, CT*
Stone, Alexander P., *University of New Mexico*
Straffin, Jr., Philip D., *Beloit College*
Sved, Marta, *University of Adelaide*
Thompson, Thomas M., *Walla Walla College*
Thuente, David J., *Indiana-Purdue University at Ft. Wayne*
Tingley, Daryl, *Univ. of New Brunswick*
Townsend, Douglas W., *Indiana-Purdue University at Ft. Wayne*
Tucker, Alan, *SUNY at Stony Brook*
Van de Wetering, R.L., *San Diego State University*
Vaughan, Theresa, *University of North Carolina*
Wagner, Clifford, *Pennsylvania State University, Harrisburg*
Wallis, W.D., *Southern Illinois Univ.*
Warner, Hoyt D., *Western New England College*
Watkins, John J., *Colorado College*
Webb, William, *Washington State University*
Weiss, Asia Ivic, *York University*
Wilker, J. B., *University of Toronto, Scarborough*
Wilson, Robin, *Colorado College*
Winkel, Brian J., *Rose Hulman Institute of Tech.*
Zerger, Monte, *Adams State College*
Zoeller, Bettina, *Western Kentucky University*
Zorn, Paul, *St. Olaf College*



CONTENTS

ARTICLES

NOTES

PROBLEMS

REVIEWS

NEWS AND LETTERS

INDEX TO VOLUME 61

THE MATHEMATICAL ASSOCIATION OF AMERICA
1529 Eighteenth Street, N.W.
Washington D.C. 20036